Qualitative properties of profit-making k-out-of-n systems subject to two kinds of failures

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ABSTRACT

This paper derives several properties of the optimal k-out-of-n systems where: 1) the i.i.d. components can be, with a pre-specified frequency, in one of two possible modes, 2) components are subject to failures in each of the two modes, and 3) the costs of two kinds of system failures are not necessarily the same. A characterization of the optimal k which maximizes the mean system-profit is obtained (a special case of this optimization criterion is the maximization of system reliability). We show how one can predict, based directly on the parameters of the system, whether the optimal k is smaller or larger than ½ n. Also, the directions of change in the optimal k resulting from changes in system parameters are ascertained. A sub-class of our formulation and results corresponds to the case examined in the literature in which the optimal k is chosen to maximize the system's reliability.

Qualitative Properties of Profit-Making k-out-of-n Systems Subject to Two Kinds of Failures

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Key Words — Two failure modes, k-out-of-n system, Different costs of different failures, Profit maximizing system, Properties of optimal k.

Reader Aids —
Purpose: Report results and advance state of the art
Special math needed for explanations: Probability
Special math needed to use results: Same
Results useful to: Reliability theoreticians and analysts.

Summary & Conclusions — This paper derives several properties of the optimal k-out-of-n systems where: 1) the i.i.d. components can be, with a pre-specified frequency, in one of two possible modes, 2) components are subject to failures in each of the two modes, and 3) the costs of two kinds of system failures are not necessarily the same. A characterization of the optimal k which maximizes the mean system-profit is obtained (a special case of this optimization criterion is the maximization of system reliability). We show how one can predict, based directly on the parameters of the system, whether the optimal k is smaller or larger than $\frac{1}{2}n$. Also, the directions of change in the optimal k resulting from changes in system parameters are ascertained. A sub-class of our formulation and results corresponds to the case examined in the literature in which the optimal k is chosen to maximize the system's reliability.

1. INTRODUCTION

A concern of reliability studies is to help understand how, using unreliable components, better systems can be designed. This paper studies the following problem. The system consists of n identically distributed and statistically independent components that can be, with a pre-specified frequency, in one of the two possible modes: mode 1 (closed) and mode 2 (open). The components are subject to failures in each mode. Thus, the two types of component failure are:

- failure in mode 1 (failure to close),
- failure in mode 2 (failure to open).

The system is closed if k-out-of-n components are closed. The two types of system failure, therefore, are:

- failure to close, which occurs if fewer than k components close when commanded to close,
- failure to open, which occurs if k or more components do not open when commanded to open.

In general, these two kinds of system failure can have quite different costs. The objective of this paper is to analyze the optimal k which maximizes the mean system-profit.

Three categories of results are obtained.

1. A characterization of the optimal k. This characterization is then employed to identify the circumstances under which one can predict, based directly on the parameters of the system, whether the optimal k is smaller than or larger than $\frac{1}{2}n$. The same characterization also predicts the effect on the optimal k of a change in the costs of the two kinds of system failure.
2. Several propositions concerning the nature of change in the optimal k from a change in n.
3. The effect on optimal k of changes in the probabilities of a component failure. Some of these results are based on an approximation in which the binomial distribution is approximated by a s-normal distribution. These results are supported with a range of numerical simulations.

A special case of the optimization criterion employed in this paper is one where:

- the costs of the two kinds of system failure are identical,
- the system is in the two modes with equal frequency.

Our optimization criterion then becomes the same as maximizing the system reliability. This special case has been studied by many authors (eg, Ansell & Bendell [1], Ben-Dov [2], Phillips [3]). By contrast, our analysis deals with cases which need not satisfy these special restrictions.

Among the common examples of the type of systems studied in this paper are relay circuits (subject to failures to energize and failures to de-energize) and monitoring safety systems (subject to failures in detecting a break-in and failures leading to a false alarm). Similar systems are also relevant in the context of economic organizations where managers' judgments (concerning, for instance, the acceptance or rejection of projects or ideas) are fallible. Sah & Stiglitz [4, 5] have studied the economic consequences of such fallibility for the structures of several kinds of organizations. For example, a committee with n members which approves a project only if k or more members approve it is analogous to a k-out-of-n:G system. Further, if $k = \frac{1}{2}n$ then the decisions of such a committee are based on commonly observed simple majority rule.

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2. NOTATION

p_1 Probability of component success in mode 1 (succeeding to close). $1 > p_1 > 0$.

p_2 Probability of component failure in mode 2 (failing to open). $1 > p_2 > 0$. $p_1 > p_2$.

q_i $q_i \equiv 1 - p_i$

$h_1(k)$ Probability of system success in mode 1 (succeeding to close).

$h_2(k)$ Probability of system failure in mode 2 (failing to open).

n number of components in the system. $n > 1$.

$h_i(k)$ $\sum_{j=k}^{n} \binom{n}{j} p_i^j q_i^{n-j} \equiv \text{binfc}(k; p_i, n)$.

$\bar{h}_i(k)$ $\bar{h}_i(k) \equiv 1 = h_i(k)$.

α probability with which the system is in mode 1. Thus, $1 - \alpha$ is the probability with which the system is in mode 2. $1 > \alpha > 0$.

B^1 gain from system success in mode 1.

B^2 gain from system failure in mode 1. $B^1 > B^2$.

B^3 gain from system success in mode 2.

B^4 gain from system failure in mode 2. $B^3 > B^4$.

β $(1 - \alpha)(B^3 - B^4)/\alpha(B^1 - B^2)$. $\beta > 0$.

k^* optimal k: the k which maximizes the mean system-profit.

Other, standard notation is given in "Information for Readers & Authors" at rear of each issue.

3. ASSUMPTIONS

1. The system consists of n i.i.d. components.
2. The components are commanded, with a prespecified frequency, to be in one of the two possible modes: closed and open.
3. In each of the two modes, the components are subject to failures (the failure to close when commanded to close, and the failure to open when commanded to open).
4. The system is closed if k or more components are closed. Thus, the system is subject to two kinds of failures (the failure to close when commanded to close, and the failure to open when commanded to open). The two kinds of system failures can have different costs.

4. GLOBAL PROPERTIES OF OPTIMAL k

The criterion for judging the performance of the system is the mean system-profit:

$$\alpha[B^1 h_1(k) + B^2 \bar{h}_1(k)] + (1-\alpha)[B^3 \bar{h}_2(k) + B^4 h_2(k)]. \quad (1)$$

This expression is to be maximized with respect to k. The maximization of (1) is same as the maximization of:

$$Y(k) \equiv h_1(k) - \beta h_2(k). \quad (2)$$

The summary parameter β can thus be viewed as the cost of system failure in mode 2, relative to the gain from system success in mode 1. A special case of (2) is $\beta = 1$. In this case, $Y(k)$ is the same as the reliability of the system, $h_1 - h_2$. The optimization of this special case has been studied by Ansell & Bendell [1], Ben-Dov [2], and Phillips [3]. In our more general analysis, it is straightforward to identify the results corresponding to this special case.

The Appendix shows that $Y(k)$ is a single-peaked function of k. Therefore an interior k (that is, where $n > k > 0$) is optimal if and only if it satisfies:

$$Y(k) - Y(k-1) \geqslant 0, \text{ and} \quad (3)$$

$$Y(k) - Y(k+1) \geqslant 0, \text{ with at least one strict inequality.} \quad (4)$$

By the same logic, $k = 0$ is optimal if and only if:

$$Y(0) \geqslant Y(1); \quad (5)$$

and $k = n$ is optimal if and only if:

$$Y(n) \geqslant Y(n-1). \quad (6)$$

Substitution of the definition of $h_i(k)$ into (3) to (6), and a rearrangement of the resulting expressions yields theorem 1.

Theorem 1

The necessary and sufficient conditions for the determination of k^* are:

$$(q_1/q_2)^{n-k^*}(p_1/p_2)^{k^*} \geqslant \beta \geqslant (q_1/q_2)^{n-k^*+1}(p_1/p_2)^{k^*-1},$$

$$\text{for } n - 1 \geqslant k^* \geqslant 1. \quad (7)$$

$$(q_1/q_2)^n \geqslant \beta, \text{ for } k^* = 0, \quad (8)$$

$$\beta \geqslant (q_1/q_2)(p_1/p_2)^{n-1}, \text{ for } k^* = n. \quad (9)$$

One of the uses of theorem 1 is that it can help predict certain properties of the magnitude of k^*, based directly on the values of the parameters β, p_1, p_2. In particular, theorem 2 delineates sufficient conditions under which k^* is smaller than, larger than, or equal to $\frac{1}{2}n$.

Theorem 2

$$k^* < \tfrac{1}{2}n + 1; \text{ if } \beta < 1, \text{ and } p_2 \leqslant q_1. \quad (10)$$

$$k^* > \tfrac{1}{2}n; \text{ if } \beta > 1, \text{ and } p_2 \geqslant q_1. \quad (11)$$

$$k^* = \tfrac{1}{2}(n+1), \text{ for odd } n;\ k^* = \tfrac{1}{2}n \text{ or } \tfrac{1}{2}n + 1,$$

$$\text{for even } n, \text{ if } \beta = 1, \text{ and } p_2 = q_1 \quad (12)$$

(See the Appendix for a proof.)

Theorem 1 also yields theorem 3 concerning the effect of β on k^*.

Theorem 3

An interior k^* is non-decreasing in β. If the increase (respectively, decrease) in β is sufficiently large, then an interior k^* must increase (respectively, decrease).

(See the Appendix for a proof.)

5. LOCAL EFFECTS OF THE PARAMETERS ON THE OPTIMAL k

This section presents some results concerning how an interior k^* is altered due to a change in the number of components, n, or due to a change in the parameters p_1 and p_2. The emphasis is on identifying qualitative aspects of these effects; for example, under what circumstances k^* increases or decreases if a particular parameter changes. The analysis is based on two analytic simplifications:

1. k is treated as a continuous variable.
2. the binomial Sf in the definition of $h_i(k)$ is approximated by

$$h_i(k) = \mathrm{gaufc}(z_i) \tag{13}$$

$$z_i \equiv (k - np_i)/[np_iq_i]^{1/2}.$$

As is well known, there are other approximations of a binomial distribution which can be more accurate than (13) for particular ranges of parameters. Also, even when (13) provides a satisfactory approximation of h_i, it might not be the case that the derivatives of h_i (to be used in the analysis below) are satisfactorily approximated when (13) is used. To support our results, therefore, a range of numerical simulations is presented at the end of the paper. Moreover, our main objective here is to derive qualitative results concerning the direction of change in k^* due to a change in parameters. These results are less likely to be sensitive to the approximation than, for instance, the results concerning the magnitude of change in k^*.

The derivative of $Y(k)$ with respect to k is:

$$Y_k(k) \equiv h_{1k}(k) - \beta h_{2k}(k) \tag{14}$$

$$h_{ik}(k) \equiv \partial h_i(k)/\partial k; \text{ for } i = 1, 2.$$

Thus, the interior extreme points of $Y(k)$ are those which satisfy:

$$Y_k(k) = 0. \tag{15}$$

Next, it is easily shown that $Y(k)$ is strictly concave in k (that is, $\partial Y_k(k)/\partial k < 0$) at any k which satisfies (15) (for a confirmation, see (A.16) in the Appendix). It follows therefore that, for an interior k^*, eq. (15) represents the necessary and sufficient condition for optimality.

Let θ represent a parameter (θ is n, p_1, or p_2), then a perturbation in (15) yields—

$$\frac{dk^*}{d\theta} = -\frac{\partial Y_k}{\partial \theta} \Big/ \frac{\partial Y_k}{\partial k}, \tag{16}$$

where the r.h.s. of (16) is evaluated at k^*; viz, at the value of k which satisfies (15). Using the above method, we obtain the following expression for dk^*/dn (see the Appendix for a derivation).

$$\frac{dk^*}{dn} = \frac{1}{2n} \cdot \frac{(k^*)^2q_1q_2 + \{n^2 - (k^*)^2\}p_1p_2}{k^*q_1q_2 + (n - k^*)p_1p_2}. \tag{17}$$

Eq. (17) yields several qualitative conclusions concerning the local change in an interior k^* as n changes. These are summarized below.

Theorem 4

$$\text{i. } 1 > \frac{dk^*}{dn} > 0. \tag{18}$$

$$\text{ii. } \frac{dk^*}{dn} \gtrless 1/2, \text{ if } p_2 \gtrless q_1. \tag{19}$$

(See the Appendix for proofs.)

Part *i* of theorem 4 has a clear meaning: k^* increases if n increases, but the increase in k^* is smaller than that in n. Part *ii* shows that dk^*/dn is larger or smaller than 1/2, depending on whether the probability of component failure in mode 2 is larger or smaller than the probability of component failure in mode 1.

The final result, theorem 5, ascertains the direction of local change in k^* from a change in the probabilities of component failure, when these probabilities are the same in the two modes.

Theorem 5

$$\frac{dk^*}{dp_1} \gtrless 0, \text{ if } \beta \lesseqgtr 1, \text{ and } p_2 = q_1. \tag{20}$$

(See the Appendix for a proof.)

This result also has the following implication. Eqs. (10)-(12) imply, in the present case where k is treated as a continuous variable, that:

$$\frac{k^*}{n} \lessgtr 1/2; \text{ if } \beta \lessgtr 1, \text{ and } p_2 = q_1. \tag{21}$$

Thus, (20) in combination with (21) implies that k^*/n becomes closer to 1/2 as the probabilities of component failure become smaller.

6. NUMERICAL SIMULATIONS

The results (18)-(20) are based on the approximation (13). To test the suitability of this approximation for these results, we undertook numerical simulations, for a range of parameters, of exact changes in k^* due to changes in parameters. The exact changes in k^* are computed as follows. For each combination of parameters, the integer value of k^* is first calculated directly from (7)-(9). One of the parameters is then altered, and the new integer value of k^* is similarly calculated.

Our simulations support, within the ranges of parameters we have considered, the results (18)-(20). The approximation (13), however, might not always be suitable for the results under consideration, if the parameters are sufficiently different from those we have considered.

The first set of simulations was for each of the following 540 combinations of parameters: n = (25, 45, 65, 85, 105), β = (0.05, 0.1, 0.75, 1.5), p_1 = (0.3, 0.35, 0.4, 0.45, 0. 5, 0.55, 0.6, 0.65, 0.7), and $p_2 = (p_1 - 0.05, p_1 - 0.1, p_1 - 0.2)$. In each case, the value of n was increased by 2 and the resulting increase in the integer value of k^* was computed. For all cases for which the pre- and post-change k^* had an interior value (which was true for 529 out of 540 cases), the change in k^* was non-negative, not larger than 2, not smaller than 1 if $p_2 \geqslant q_1$, and not larger than 1 if $p_2 \leqslant q_1$. These results are consistent with (18) and (19) in which, it will be recalled, n and k^* were treated as continuous variables.

The second set of simulations was for each of the following 64 combinations of parameters: n = (25, 50, 75, 100), β = (0.05, 0.1, 0.75, 1.5), p_1 = (0.55, 0.6, 0.65, 0.7) and $p_2 = q_1$. For all cases, the change in k^* was calculated by increasing p_1 by 0.05. As (20) suggests, this change in k^* was non-negative for $\beta \leqslant 1$, and non-positive for $\beta \geqslant 1$.

APPENDIX

Proof that $Y(k)$ is a single-peaked function of k

From the definition of $h_i(k)$, we obtain:

$$h_i(k) - h_i(k-1) = -\binom{n}{k-1} p_i^{k-1} q_i^{n-k+1}. \tag{A.1}$$

Eqs. (2) and (A.1) yield:

$$Y(k) - Y(k-1) = \binom{n}{k-1} [-p_1^{k-1} q_1^{n-k+1} + \beta p_2^{k-1} q_2^{n-k+1}]. \tag{A.2}$$

$$Y(k+1) - Y(k) = \binom{n}{k} [-p_1^k q_1^{n-k} + \beta p_2^k q_2^{n-k}]. \tag{A.3}$$

Using (A.3), we can rewrite (A.2) as:

$$Y(k) - Y(k-1) = [Y(k+1) - Y(k)]a_1(k) + a_2(k), \tag{A.4}$$

where

$$a_1(k) \equiv q_1 \binom{n}{k-1} / p_1 \binom{n}{k} > 0,$$

$$a_2(k) \equiv \binom{n}{k-1} \beta p_2^k q_2^{n-k} (q_1/p_1) \left[\frac{p_1 q_2}{p_2 q_1} - 1\right] > 0.$$

The sign of $a_1(k)$ is obvious. The sign of $a_2(k)$ is established by noting that $(p_1q_2)/(p_2q_1) > 1$ because $p_1 > p_2$.

Eq. (A.4) implies that:

$$Y(k-1) - Y(k-2) = [Y(k) - Y(k-1)] a_1(k-1) + a_2(k-1).$$

Since $a_1(k-1)$ and $a_2(k-1)$ are both positive, it follows that:

$$\text{If } Y(k) \geqslant Y(k-1), \text{then } Y(k-1) > Y(k-2). \tag{A.5}$$

Another implication of (A.4) is that:

$$Y(k+1) - Y(k) = [Y(k+2) - Y(k+1)] a_1(k+1) + a_2(k+1).$$

Since $a_1(k+1)$ and $a_2(k+1)$ are both positive, it follows that:

$$\text{If } Y(k) \geqslant Y(k+1), \text{then } Y(k+1) > Y(k+2). \tag{A.6}$$

Eqs. (A.5) and (A.6) show that $Y(k)$ is single-peaked in k.

Proof of Theorem 2

For brevity, we use the symbols:

$\phi \equiv p_1/p_2$, and $\rho \equiv q_1/q_2$.

Then, since $p_1 > p_2$, it is easily established that

$$\rho\phi \gtrless 1, \text{if } p_2 \lessgtr q_1. \tag{A.7}$$

Eq. (7) can be rewritten as:

$$(n - k^*)\ln(\rho\phi) + (2k^* - n)\ln(\phi) \geqslant \ln(\beta) \geqslant (n - k^* + 1)\ln(\rho\phi) + (2k^* - n - 2)\ln(\phi). \tag{A.8}$$

Now suppose (10) is not true. That is, $k \geqslant \frac{1}{2}n + 1$, when $\beta < 1$, and $p_2 \leqslant q_1$. Then, the r.h.s. of (A.8) is nonnegative because $\rho\phi \geqslant 1$ from (A.7), $n - k^* + 1 > 0$, $\phi > 1$, and $(2k^* - n - 2) \geqslant 0$. On the other hand, $\ln(\beta) < 0$. Expression (A.8) is thus contradicted. An analogous argument shows that (A.8) is contradicted if (11) is not true.

Finally consider the case where $\beta = 1$, and $p_2 = q_1$. Then $\rho\phi = 1$ from (A.7). Thus (A.8) becomes—

$$(2k^* - n)\ln(\phi) \geq 0 \geq (2k^* - n - 2)\ln(\phi). \qquad (A.9)$$

Now since $\ln(\phi) > 0$, it follows from (A.9) that $k^* = ½(n + 1)$ if n is odd. If n is even, then (A.9) is satisfied for either $k^* = ½n$, or $k^* = ½n + 1$.

Proof of Theorem 3

Use symbols ρ and ϕ defined in the proof of theorem 2; eq. (7) can be rewritten as —

$$(\phi/\rho)^{k^*} \geq \beta/\rho^n \geq (\phi/\rho)^{k^*-1}, \qquad (A.10)$$

where $\phi/\rho = (p_1q_2)/(p_2q_1) > 1$ because $p_1 > p_2$. Now suppose β is changed to β' such that $\beta' > \beta$. If the corresponding optimal k is denoted by k'^*, then—

$$(\phi/\rho)^{k'^*} \geq \beta'/\rho^n \geq (\phi/\rho)^{k'^*-1}. \qquad (A.11)$$

Since $\beta' > \beta$, the second part of the inequality (A.10) yields: $\beta'/\rho^n > (\phi/\rho)^{k^*-1}$. The preceding equation implies that the first part of the inequality (A.11) will be contradicted if $k'^* \leq k^* - 1$. Therefore $k'^* \geq k^*$. Next, if β' is sufficiently larger than β (specifically, if $\beta'/\rho^n > (\phi/\rho^{k^*}) \geq \beta/\rho^n$) then it is obvious that (A.11) will be satisfied only if $k'^* > k^*$.

Expression for dk^*/dn

For brevity in the derivations below, we suppress the arguments of functions Y_k, h_{ik}, and z_i. Differentiation of (13) with respect to k yields —

$$h_{ik} = -\text{gaud}(z_i)[np_iq_i)]^{-1/2} < 0, \qquad (A.12)$$

because gaud $(z_i) > 0$ for all z_i.

Next, Y_k in (14) can be rewritten as $Y_k = h_{2k}[(h_{1k}/h_{2k}) - \beta]$. Thus, the derivative $\partial Y_k/\partial k$, evaluated at $Y_k = 0$, is:

$$\partial Y_k/\partial k = h_{2k}\frac{\partial}{\partial k}(h_{1k}/h_{2k}). \qquad (A.13)$$

Using (A.13) and (A.12), one obtains —

$$\partial Y_k/\partial k = bh_{1k}(p_1 - p_2)/(np_1p_2q_1q_2), \qquad (A.14)$$

$$b \equiv k^*q_1q_2 + (n - k^*)p_1p_2. \qquad (A.15)$$

It follows from (A.14) that —

$$\partial Y_k/\partial k < 0, \qquad (A.16)$$

because $b > 0$, $h_{1k} < 0$, and $p_1 > p_2$.

Analogous to (A.13), the expression for $\partial Y_k/\partial \theta$, evaluated at k^*, is —

$$\partial Y_k/\partial \theta = h_{2k}\frac{\partial}{\partial \theta}(h_{1k}/h_{2k}). \qquad (A.17)$$

If θ is n, then (A.17) and (A.12) yield—

$$\partial Y_k/\partial n = -h_{1k}(p_1 - p_2)[(k^*)^2(1 - p_1 - p_2) + n^2p_1p_2]/(2n^2p_1p_2q_1q_2). \qquad (A.18)$$

Substitute (A.14) and (A.18) into (16), and rearrange; one obtains—

$$\frac{dk^*}{dn} = [(k^*)^2q_1q_2 + \{n^2 - (k^*)^2\}p_1p_2]/(2nb). \qquad (A.19)$$

Recall the definition of b in (A.15). Then (A.19) is the same as (17).

Proof of Theorem 4

Expression (A.19) is obviously positive. It also yields

$$1 - \frac{dk^*}{dn} = [k^*(2n - k^*)q_1q_2 + (n - k^*)^2p_1p_2]/(2nb) > 0. \qquad (A.20)$$

This establishes part i. Eq. (A.21) is obtained from (A.19).

$$\frac{dk^*}{dn} - ½ = k^*(k^* - n)(1 - p_1 - p_2)/(2nb). \qquad (A.21)$$

Part ii follows immediately from (A.21).

Proof of Theorem 5

Under the assumption that $p_2 = q_1$, (A.12) and (A.17) yield the following derivative for $\theta = p_1$

$$\partial Y_k/\partial p_1 = -h_{1k}(n - 2k^*)[q_1^2 + p_1^2]/(2p_1^2q_1^2) \qquad (A.22)$$

Using (16), (A.16) and (A.22), it follows that—

$$\frac{dk^*}{dp_1} \gtrless 0, \text{ if } \frac{k^*}{n} \lessgtr ½. \qquad (A.23)$$

Next, since k is being treated as a continuous variable, (10)-(12) imply—

$$\frac{k^*}{n} \lessgtr ½, \text{ if } \beta \lessgtr 1 \text{ and } p_2 = q_1. \qquad (A.24)$$

Combination of the above with (A.23) yields (20).

REFERENCES

[1] J. Ansell, A. Bendell, "On the optimality of k-out-of-n systems", *IEEE Trans. Reliability*, vol R-31, 1982 Jun, pp 206-210.

[2] Y. Ben-Dov, "Optimal reliability design of k-out-of-n systems subject to two kinds of failure", *J. Opl. Res. Soc.*, vol 31, 1980, pp 743-748.

[3] M. J. Phillips, "k-out-of-n:G systems are preferable", *IEEE Trans. Reliability*, vol R-29, 1980 Jun, pp 166-169.

[4] R. K. Sah, J. E. Stiglitz, "The Architecture of Economic Systems: Hierarchies and Polyarchies", *American Economic Review*, vol 76, 1986 Sep, pp 716-727.

[5] R. K. Sah, J. E. Stiglitz, *Human Fallibility and Economic Organizations*, forthcoming, MIT Press, 1988.

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