The effects of child mortality changes on fertility choice and parental welfare

Raaj K. Sah

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ABSTRACT

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The Effects of Child Mortality Changes on Fertility Choice and Parental Welfare

Raaj K. Sah

University of Chicago

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I. Introduction

An ongoing concern in the economic study of population has been to understand the effects of changes in child mortality rates on parental fertility decisions. This concern stems partly from the set of unprecedented transitions in child mortality rates and parental fertility choices that have been, and continue to be, observed at different times in different countries. Empirical studies have overwhelmingly shown that the number of children produced by a couple declines as the mortality rate declines.[1] This relationship also appears quite

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[1] Schultz (1981) provides a review of the economic literature. Freedman (1975) provides a summary and a bibliography of the demographic and sociological literature. A collection of demographic essays focusing on the effects of child mortality is found in

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intuitive.[2] Yet it has not been possible to satisfactorily analyze this relationship in even the simplest theoretical models of fertility choice. This paper attempts to bridge this gap between theory and the empirical literature.

Consider the following bare-bones model. A couple makes a one-time decision concerning the number of births. This decision maximizes the expected utility, inclusive of all benefits and costs, from different numbers of surviving children. Even when such a simple single-stage choice model is used and additional strong assumptions are made, what has typically been shown is that a decline in the mortality rate can lead, with equal plausibility, to an increase or a decrease in fertility.

Among the assumptions that previous theoretical studies dealing with this issue have had to make are the following: (i) The expected utility depends on the expected number of surviving children (see Ben-Porath and Welch 1972; Ben-Porath 1976). As is explained later, this assumption either is inconsistent with parental choice under uncertainty or predicts an outcome that contradicts the pattern that is typically observed. (ii) The ex post utility is quadratic in the number of surviving children.[3] The limitations of the quadratic assumption are well known; there are no special reasons that make this assumption any less limiting in the fertility context. (iii) Only the polar outcomes matter, that is, when all children who are born survive or when none survives. The utilities of other outcomes, when some but not all children die, do not matter. As is described later, O'Hara (1975) uses this assumption in his analysis of fertility.

Since the theoretical predictability established to date is so weak in a model as simple as the single-stage model noted above, it is not surprising that this predictability is no better in more general models. An important aspect of fertility decisions is that they are dynamic and stochastic. Past fertility choices stochastically influence, through mortality, the number and the age composition of children currently

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Preston (1978). Scores of empirical studies cited in these three references show that a lower mortality rate lowers the number of births. There are some exceptions as well. Dyson and Murphy (1985) note that, in some cases, a decline in the mortality rate might have been accompanied by a brief increase in the economywide fertility, primarily because of such contemporaneous changes as a decrease in widowhood and disease-related sterility. However, this increase was quickly overwhelmed by the direct effect of a lower child mortality rate, namely, a rapid decrease in the fertility of couples. The present paper focuses on the direct effect. Note that, throughout this paper, "mortality" refers to child mortality.

[2] See Becker's pioneering paper on the economics of fertility (Becker 1960, p. 212).

[3] See, e.g., Newman (1988). His analysis of the effect of mortality focuses on the marginal replacement behavior (i.e., on the change in fertility due to one extra death of a child) rather than on the expected number of total births.

alive. This, in turn, influences current and future fertility decisions. Thus fertility choice can be naturally formulated as a dynamic stochastic program.[4]

This paper presents some results on the effects of mortality changes on fertility. Using a single-stage choice model, I first show that some simple and plausible conditions are sufficient to yield the typically observed pattern that the number of births declines if the mortality rate declines. Then, using a two-stage dynamic stochastic model of fertility choice, I examine the effects of a decline in mortality rates on the number of births in each of the two periods, as well as on the expected number of total births. I also show how this analysis extends to a multistage dynamic stochastic model.

Another concern of this paper is to examine the effects of mortality changes on parental welfare. The assessment of such welfare effects does not appear to have received attention in the literature, even though it is a necessary component of an economic evaluation of government programs aimed at reducing child mortality. I show, for instance, that a lower mortality rate raises parental welfare. This intuitive result is, to my knowledge, new. Also, the welfare results presented here are robust. For example, they do not depend on the properties of the utility function: they are a consequence primarily of the optimizing behavior. At the same time, the results are not obvious; for example, they cannot be obtained from the envelope theorem or stochastic dominance arguments alone.

In this paper, the number of children, born or surviving, is represented as a nonnegative integer. This is obviously the correct representation of reality. Also, a continuous representation of intrinsically discrete variables may be a greater source of error in the context of fertility choice (because, in some cases, none or only one of the children born may survive) than in many other contexts (such as a factory producing millions of widgets) in which the variables have large values. In contrast, most previous theoretical studies have employed a continuous representation, perhaps because of its seeming tractability. It turns out, however, that a discrete representation yields a crisper analysis. Moreover, for reasons discussed later, this paper's results cannot easily be obtained, using assumptions comparable to those that I have used, if a continuous representation is employed. For the problem at hand, therefore, realism and tractability coincide. Also, this paper's methodology for analyzing discrete choices may be

[4] See Heckman and Willis (1976) for an early formulation and empirical demonstration of this approach. See Wolpin (1984) for an estimation that emphasizes child mortality and also for a useful discussion of some of the issues faced in the empirical implementation of this approach.

useful in examining other fertility-related questions (e.g., the fertility impact of parental preferences for children's gender) as well as other economic problems that entail discrete choices.

To focus on the fertility choice, other parental choices (e.g., child care and quality, parental human capital formation, and labor market participation) are kept in the background by assuming that these choices are made optimally for every fertility choice. Consequently, the results presented in this paper hold if these other choices and the associated budget constraints are analyzed simultaneously along with the fertility choice.

A single-stage model of fertility choice is analyzed in Section II. Section III analyzes a two-stage model and discusses its extension to a multistage model. Section IV presents some concluding remarks. The specific aim of this paper is to use simple but realistic models to extract some predictability concerning the effects of mortality changes on fertility choice and parental welfare. If no predictability can be established in simpler models, it is unlikely to be established in more general models. On the other hand, if some predictability can be established in simpler models, which turns out to be the case here, then the same approach may be useful in other models.

II. A Single-Stage Model of Fertility Choice

The number of children produced by a couple is denoted by the integer variable n. The random variable N denotes the number of children who will survive, with $n \geq N \geq 0$. I assume that the survival of each child is an independent event with probability s, with $1 > s > 0$. The mortality rate thus is $1 - s$. Consequently, a larger value of s represents a regime of lower mortality.[5] Also, the probability that N out of n children will survive is the binomial density

$$b(N, n, s) \equiv \binom{n}{N} s^N (1 - s)^{n-N}. \tag{1}$$

[5] Several modifications of this aspect are possible. One is to treat the survival of the ith child as an independent event with probability s_i. In this case, a regime of lower mortality may be represented by positing that s_i changes to $s_i + \theta_i$, where $\theta_i \geq 0$ for all i and $\theta_i > 0$ for at least one i. However, such a distinction among children may not be appropriate within the single-stage model under consideration here, where the underlying simplification is that all births take place at the same time and that all deaths take place at the same time in the future. If one wishes to highlight ex ante distinctions among children, then it is perhaps better to use a dynamic framework, as is done in the next section. Another possible modification is to let s depend on the number of births. In this case, a regime of lower mortality may be represented by positing that $s(n)$ changes to $s(n) + \theta(n)$, where $\theta(n) \geq 0$ for all n and $\theta(n) > 0$ for at least one n. Given the objective of the paper, described at the end of the last section, I shall not deal with such modifications.

It is assumed that the mortality rate is an exogenous parameter to the parents, who are being considered in isolation from the rest of the economy. However, the analysis remains unaffected if, instead, the mortality rate is endogenously determined by the parents through a production function. In this case, one would examine the fertility effects of a change in parameters (e.g., preventive health technology or the prices of relevant inputs) that could potentially alter the mortality rate. Also, though the present analysis uses the number of births as the choice variable, it can be restated using the conception probability as the choice variable.

I next describe the expected utility from different numbers of births. It is convenient to begin by considering the ex post benefits and costs and by temporarily abstracting from those ex ante benefits and costs (such as childbearing costs) that depend on the number of births, but not on how many of them survive. These ex ante benefits and costs are incorporated in the last part of this section. Let $u(N)$ denote the ex post net utility, inclusive of all benefits and costs, from N surviving children. One would expect $u(N)$ to first increase and then decrease with N. To see this, one may write $u(N)$ as $u(N) \equiv w(g(N) - h(N))$, where $g(N)$ denotes the benefit from N surviving children expressed in terms of a numeraire (say dollars), $h(N)$ denotes the corresponding cost, and the function w translates the net benefit into net utility. The standard assumption concerning the benefit $g(N)$ is that it is increasing and concave, if not strictly concave, in N. The cost $h(N)$ includes expenditures on children as well as the imputed value of household inputs (such as parents' time) that are available in limited supply and that cannot be adequately substituted by inputs bought from the market. The importance of such aspects of postbirth costs has been pointed out in the literature (see Schultz [1976, pp. 102–4; 1988, pp. 424–37] and references therein). It is thus appropriate to assume that $h(N)$ is increasing and strictly convex in N. Next, under the assumption that the parents are risk averse or risk neutral (i.e., w is concave in its argument), it follows that $u(N)$ is strictly concave in N. We assume this property of $u(N)$, although, as we shall see, it can be weakened.

The expected utility from n births, for a given s, is denoted by $U(n, s)$. Thus

$$U(n, s) \equiv \sum_{N} b(N, n, s)u(N). \tag{2}$$

Throughout the paper, we suppress the range of the index in a summation if the summation is taken over the entire range. For instance, in the right-hand side of (2), the summation is taken over $N = 0$ to n. Let $n(s)$ denote the largest optimal value of n. Let the indirect utility $V(s)$ describe the parental welfare level. Then[6]

$$V(s) \equiv \max_{n} U(n, s) \equiv U(n(s), s). \qquad (3)$$

Denote the marginal utility of a surviving child by $u_N(N) \equiv u(N + 1) - u(N)$ and the change in this marginal utility due to one more surviving child by $u_{NN}(N) \equiv u_N(N + 1) - u_N(N)$. Denote the marginal expected utility of an additional birth by $U_n(n, s) \equiv U(n + 1, s) - U(n, s)$ and the change in this marginal expected utility due to one more birth by $U_{nn}(n, s) \equiv U_n(n + 1, s) - U_n(n, s)$. These are the discrete equivalents of the first and the second partial derivatives.

The following relationships, established in the Appendix, play a central role in the analysis:

$$U_n(n, s) = s \sum_{N} b(N, n, s) u_N(N), \qquad (4)$$

$$U_{nn}(n, s) = s^2 \sum_{N} b(N, n, s) u_{NN}(N), \qquad (5)$$

$$\frac{\partial}{\partial s} U_n(n, s) = \frac{1}{s} [U_n(n, s) + nU_{nn}(n - 1, s)], \qquad (6)$$

$$\frac{\partial}{\partial s} U(n, s) = \frac{n}{s} U_n(n - 1, s). \qquad (7)$$

For an intuitive interpretation of expression (4), compare the expected utility under two alternatives: $n + 1$ births versus n births. Consider $n + 1$ states of the world in which $N = 0, 1, \ldots, n$ children out of n births survive. Now if the $(n + 1)$st child does not survive, then the parents have the same utility in each of the states under the two alternatives. If the $(n + 1)$st child survives, then the parents have one more child in each of the states under the first alternative than in the second. The resulting difference in the utility, summed over all states, is $\Sigma_N b(N, n, s) u_N(N)$. Thus since s is the probability that the $(n + 1)$st child will survive, the marginal expected utility of the $(n + 1)$st birth is given by (4).

[6] As was remarked earlier, this formulation subsumes choices other than the fertility choice. For instance, let the vector $\mathbf{x}$ denote other choices. The decision problem is $\max_{\mathbf{x},n} A(\mathbf{x}) + \Sigma_N b(N, n, s) a(N, \mathbf{x})$, where $A(\mathbf{x})$ is the part of the ex ante utility that depends only on $\mathbf{x}$ and $a(N, \mathbf{x})$ is the ex post utility that depends on $\mathbf{x}$ as well as on N. Then, defining $u(N) \equiv \max_{\mathbf{x}} A(\mathbf{x}) + a(N, \mathbf{x})$ as the maximized value of the ex post utility for each N, we get formulation (2).

Properties of the Optimal Choice

Recall that $u(N)$ is strictly concave in N; that is, $u_{NN}(N) < 0$. Thus (5) yields

$$U_{nn}(n, s) < 0. \tag{8}$$

In other words, the expected utility U is strictly concave in the number of births, n. In turn, this yields the following proposition.

PROPOSITION 1. Either the optimal number of births is unique or there are two neighboring numbers that are both optimal.

To prove this result, recall that $n(s)$ denotes the largest optimal value of n. Thus it must satisfy

$$U_n(n(s) - 1, s) \geq 0 \tag{9a}$$

and

$$U_n(n(s), s) < 0. \tag{9b}$$

From (8) and the definition of $U_{nn}(n, s)$, it follows that (9a) and (9b) yield, respectively,

$$U_n(n, s) > 0 \quad \text{if } n < n(s) - 1; \quad U_n(n, s) < 0 \quad \text{if } n \geq n(s). \tag{10}$$

Now if the inequality in (9a) is strict, then (10) shows that $n(s)$ is the unique optimal choice. Otherwise (10) shows that $n(s)$ and $n(s) - 1$ are the only two optimal choices. Note that, for brevity, these and some other results below are proved for interior values of $n(s)$, that is, for $n(s) \geq 1$.

The Effect of a Change in the Mortality Rate on the Number of Births

I first show that a lower mortality rate does not increase the number of births. That is,

$$n(s') \leq n(s) \quad \text{for } s' > s. \tag{11}$$

This result can be established as follows. Since expression (6) holds for any fixed n, we may evaluate it at $n = n(s)$. From (8) and (9b), the evaluation of (6) yields

$$\frac{\partial}{\partial s} U_n(n = n(s), s) < 0, \tag{12}$$

where $n = n(s)$ indicates that the value of n is kept unchanged at $n(s)$ in the computation of this derivative. Now let s' denote a survival probability slightly larger than s. Then (12) implies that $U_n(n(s), s') < U_n(n(s), s)$. In turn, from (9b),

$$U_n(n = n(s), s') < 0. \tag{13}$$

From (8), U_n is decreasing in n. Thus it follows from (13) that

$$U_n(n, s') < 0 \quad \text{for } n \geq n(s). \tag{14}$$

From (14), $U(n(s), s') > U(n(s) + 1, s') > \ldots$. Consequently, a value of n larger than $n(s)$ is not optimal at s'. I have thus shown that $n(s)$ is locally nonincreasing in s. The global counterpart of this result is (11), and it is obtained from the local result by using standard continuity arguments (for a proof, see a more detailed version of this paper [Sah 1990]).

Next, I rule out the uninteresting case in which the number of births remains entirely unchanged throughout the range of mortality rates. Thus (11) yields the following proposition.

PROPOSITION 2. The number of births is an increasing integer function of the mortality rate.

The sole assumption concerning the ex post utility $u(N)$ employed in the analysis above is that it is strictly concave in N. Now suppose that the ex post utility does not have this property. Even then, proposition 2 holds for local changes in the mortality rate. To see this, begin with $n(s)$, which is the largest optimal choice at s. Now consider all those slightly larger survival probabilities s' for which the optimal choice changes at most by one. That is, the candidates for the optimal choice at s' are $n(s) - 1$, $n(s)$, and $n(s) + 1$. Then

$$n(s) + 1 \text{ is not optimal at } s'. \tag{15}$$

To prove (15), the definition of $U_{nn}(n, s)$ can be used to rewrite (6) as

$$\frac{\partial}{\partial s} U_n(n, s) = \frac{1}{s}[(n + 1)U_n(n, s) - nU_n(n - 1, s)]. \tag{16}$$

Expressions (9a), (9b), and (16) imply that $\partial U_n(n = n(s), s)/\partial s < 0$. Therefore, $U_n(n(s), s') < U_n(n(s), s)$. The preceding expression and (9b) imply that $U_n(n(s), s') < 0$. This yields (15).

The Effect of a Change in the Mortality Rate on Parental Welfare

PROPOSITION 3. Parental welfare does not decrease if the mortality rate decreases. The welfare strictly increases if a decrease in the mortality rate induces a nontrivial change in the fertility choice.

To prove this result, note from (7) and (9a) that $\partial U(n = n(s), s)/\partial s \geq 0$. Thus

$$U(n(s), s') \geq U(n(s), s) \tag{17}$$

for a value of s' slightly larger than s. Also, definition (3) of the optimum implies

$$U(n(s'), s') \geq U(n(s), s'). \tag{18}$$

Expressions (3), (17), and (18) yield $V(s') \geq V(s)$. That is, the parental welfare level, V, is locally nondecreasing in s. This result has an intuitive interpretation. Expression (17) shows that parents would be no worse off at a lower mortality rate if they were to make the same choice that was optimal at a higher mortality rate. Thus their actual welfare, corresponding to the optimal choice at a lower mortality rate, cannot be lower.

A stronger result is obtained if the choices are nontrivially different at two mortality rates. Here, by a nontrivial difference I mean that a choice that is optimal at s is not optimal at s'. In this case, inequality (18) is strict. Thus it follows that parental welfare is locally increasing in s. Proposition 3 states the global counterparts of the local results just proved. Once again, the global results are obtained by using continuity arguments.

The welfare results presented above are robust because they are a consequence of the optimizing behavior, and they do not depend on the properties of the utility function. At the same time, the results are not obvious because, for instance, they require the relationship in (7).

An extension of this welfare analysis yields the magnitude of the parental gain from a decrease in the mortality rate. For example, let M denote the current parental income, and let ΔM denote a hypothetical increase in this income that has the same value to the parents as an increase in the survival probability from s to s'. We know from proposition 3 that ΔM is not negative and that it will typically be positive. Further, given any particular specification of the parental preferences and characteristics, the value of ΔM can be calculated from the equality $V(s, M + \Delta M) = V(s', M)$. This same approach is useful in models in which the parents determine the mortality rate endogenously and in which we wish to assess the magnitude of the parental gain from a change in a parameter such as preventive health technology.

Such welfare assessments can be useful. For example, governments often undertake programs aimed at reducing child mortality. Such programs are common in most developing countries, but they have also been undertaken in some developed countries for social groups experiencing high child mortality. A necessary component in an economic evaluation of such programs is the assessment of welfare gains of the type described above.

Ex Ante Costs

To incorporate the ex ante benefits and costs of different numbers of births, let $C(n)$ denote the net utility cost that depends on the number of births, n, but not on how many of them survive, N. Then (3) is replaced by $V'(s) \equiv \max_n U'(n, s)$, where

$$U'(n, s) = U(n, s) - C(n) \tag{19}$$

and U is given by (2). Define $U'_n(n, s) \equiv U'(n + 1, s) - U'(n, s)$ and $C_n(n) \equiv C(n + 1) - C(n)$. Define U'_{nn} and C_{nn} accordingly. Assume that $C_n \geq 0$ and $C_{nn} \geq 0$; that is, the marginal ex ante utility cost of births is nonnegative and nondecreasing in the number of births. From (19),

$$U'_{nn}(n, s) = U_{nn}(n, s) - C_{nn}(n), \tag{20}$$

where U_{nn} is as given by (5). The optimality conditions, (9a) and (9b), are now $U'_n(n(s) - 1, s) \geq 0 > U'_n(n(s), s)$. However, $\partial U'_n/\partial s$ and $\partial U'/\partial s$ continue to be described by the right-hand sides of (6) and (7), respectively.

It is then straightforward to verify that propositions 1 and 3 remain unaffected. To examine proposition 2, we need some additional analysis. First, if

$$\frac{\partial}{\partial s} U'_n(n = n(s), s) \leq 0, \tag{21}$$

then by going through the steps following expression (12), we can verify that (11) holds and that proposition 2 holds. To evaluate the left-hand side of (21), we derive the following in the Appendix:

$$\frac{\partial}{\partial s} U'_n(n, s) = \sum_{N=0}^{n-1} \phi(N, n - 1, s) u_{NN}(N) + u_N(n), \tag{22}$$

where

$$\phi(N, n - 1, s) \equiv (n + 1)sb(N, n - 1, s) - B(N, n - 1, s), \tag{23}$$

and $B(N, n, s) \equiv \Sigma_{j=0}^{N} b(j, n, s)$ denotes the cumulative probability of N or fewer survivals out of n births. To evaluate the sign of (22), we need the signs of the ϕ, defined in (23). These signs can be assessed numerically for the limited range of possible values of n and s that are relevant in most fertility contexts. Alternatively, they can be assessed analytically. For example, I show in the Appendix that ϕ is positive for the relevant values of N if $n \leq 12$ and $s > .81$. A survival probability smaller than .81 and a total number of births larger than 12 are not particularly relevant in many fertility contexts. Thus, for brevity, I treat the ϕ as positive in the analysis below. However, analogous results can be obtained under conditions on n and s weaker than those just noted.

Since $u_{NN}(N) < 0$ and the ϕ are positive, it follows from (22) that (21) is satisfied if $u_N(n) \leq 0$. That is, a sufficient condition for proposition 2 to hold is that the marginal ex post utility from an extra surviving child would be nonpositive if all the children from an optimally chosen number of births were to survive. Another sufficient condition for the same result can be obtained: Note from (22) that (21) is satisfied even if $u_N(n)$ is positive, provided that the $u_{NN}(N)$ have sufficiently large magnitudes in comparison to the magnitude of $u_N(n)$. In other words, proposition 2 holds if the ex post utility $u(N)$ is, in this sense, sufficiently concave in N.

Finally, in a more general formulation of the model analyzed in this subsection, we would replace the $u(N)$ in the right-hand side of (2) by a function $u'(n, N)$. The separable special case of this function is $u'(n, N) \equiv u(N) - C(n)$, which yields (19). For the more general formulation, we can obtain results analogous to those presented above. For instance, consider proposition 3. The cost of an extra birth is now defined as $-[u'(n + 1, N) - u'(n, N)]$, which depends on the realized value of N. If this cost is assumed to be nonnegative, then it is easily verified that proposition 3 remains unaffected.

III. A Dynamic Stochastic Model

As noted earlier, fertility choice can be naturally formulated as a dynamic stochastic program. This section examines a two-stage model that captures this aspect. A multistage model is then discussed. One would not expect the results in a dynamic model to be as crisp as those in the preceding section. Yet, as we shall see, a significant part of the earlier analysis carries over.

A Two-Stage Model

Since the fertility choice is being made in the present model at two different points in time, we need to assess the effect of a change in the mortality rate on three different but related fertility variables: (i) the number of births in the first period, (ii) the number of births in the second period (this is a random number, in general, because it depends on the number of surviving children from the first period), and (iii) the expected number of total births.

Let the integer variables n_1 and n_2 denote the numbers of births in the two periods. The corresponding numbers of surviving children are denoted by the random variables N_1 and N_2, with $n_1 \geq N_1 \geq 0$ and $n_2 \geq N_2 \geq 0$. The parents observe N_1 before choosing n_2. The net ex post utility, after N_2 is observed, is denoted by $u(N_1, N_2)$. This and other utilities discussed later subsume intertemporal discounting. Let s_1 and s_2 denote the survival probabilities of a child born in the first and second periods, respectively. A special case of this model applies when s_1 equals s_2, but this special case does not particularly simplify the analysis. If the parental welfare level is described by the indirect utility $V^1(s_1, s_2)$, then

$$V^1(s_1, s_2) \equiv \max_{n1} \sum_{N1} b(N_1, n_1, s_1) \max_{n2} \sum_{N2} b(N_2, n_2, s_2) u(N_1, N_2). \quad (24)$$

Once again, this model is highly simplified. It abstracts from those costs that depend on the number of births but not on the number of surviving children, from the interim benefits or costs of the children surviving from the first period (before the outcome of the second-period choice is known), and from the potential fertility effects of possible deaths of the first-period children after the second-period choice has been made. These aspects are discussed later.

Let u_1 and u_2 denote, respectively, the marginal utilities of a surviving child from the two periods; that is, $u_1(N_1, N_2) \equiv u(N_1 + 1, N_2) - u(N_1, N_2)$, and u_2 is defined similarly. Denote the changes in these marginal utilities due to one more surviving child by $u_{11}(N_1, N_2) \equiv u_1(N_1 + 1, N_2) - u_1(N_1, N_2)$ and by u_{12} and u_{22}, defined similarly. I assume that the surviving children from the two periods are substitutes (i.e., $u_{12} \equiv u_{21} < 0$) and that u is strictly concave in N_2 (i.e., $u_{22} < 0$).

Choice in the Second Period

We begin by considering the parental choice after N_1 has been observed. Define

$$U^2(N_1, n_2, s_2) \equiv \sum_{N2} b(N_2, n_2, s_2) u(N_1, N_2). \quad (25)$$

Let $n_2(N_1, s_2)$ denote the largest optimal value of n_2. Denote the maximized value of U^2 by

$$V^2(N_1, s_2) \equiv \max_{n2} U^2(N_1, n_2, s_2) \equiv U^2(N_1, n_2(N_1, s_2), s_2). \quad (26)$$

This choice problem is similar to the single-stage problem in the last section, except that the present problem is parameterized by the observed value of N_1. Define $U_n^2(N_1, n_2, s_2) \equiv U^2(N_1, n_2 + 1, s_2) -$

$U^2(N_1, n_2, s_2)$. Define U^2_{nn} accordingly. Then, analogous to (5), (6), (7), and (8), we obtain

$$U^2_{nn}(N_1, n_2, s_2) = (s_2)^2 \sum_{N_2} b(N_2, n_2, s_2) u_{22}(N_1, N_2) < 0, \tag{27}$$

$$\frac{\partial}{\partial s_2} U^2_n(N_1, n_2, s_2) = \frac{1}{s_2}[U^2_n(N_1, n_2, s_2) + n_2 U^2_{nn}(N_1, n_2 - 1, s_2)], \tag{28}$$

and

$$\frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2) = \frac{n_2}{s_2} U^2_n(N_1, n_2 - 1, s_2). \tag{29}$$

Also, similar to (9a) and (9b), an optimum is characterized by

$$U^2_n(N_1, n_2(N_1, s_2) - 1, s_2) \geq 0 > U^2_n(N_1, n_2(N_1, s_2), s_2). \tag{30}$$

Since the analysis of this optimum closely follows that of the single-stage problem in the last section, I only summarize the following results, which are the analogues of propositions 1, 2, and 3, respectively.

PROPOSITION 4. (i) Either the optimal number of births in the second period is unique or there are two neighboring numbers that are both optimal. (ii) The number of births in the second period is an increasing integer function of the mortality rate of a second-period child. (iii) If $s_2' > s_2$, then $V^2(N_1, s_2') \geq V^2(N_1, s_2)$. This inequality is strict if a change in the mortality rate of a second-period child alters nontrivially the second-period fertility choice.

The Effect of the Number of Surviving Children from the First Period on the Number of Births in the Second Period

A new question that arises in the present model is, How does N_1 affect $n_2(N_1, s_2)$? To ascertain this effect, the following expression is established in the Appendix:

$$U^2_n(N_1 + 1, n_2, s_2) < U^2_n(N_1, n_2, s_2). \tag{31}$$

Using expressions (27), (31), and the second part of (30), we obtain

$$U^2_n(N_1 + 1, n_2, s_2) < 0 \quad \text{for } n_2 \geq n_2(N_1, s_2). \tag{32}$$

From (32), it follows that a value of n_2 larger than $n_2(N_1, s_2)$ is not optimal for $N_1 + 1$. In other words, $n_2(N_1, s_2)$ is nonincreasing in N_1. Next, we rule out the case in which the optimal n_2 is entirely unaffected by N_1. In this uninteresting case, fertility choice is completely separable between the periods; the second period's choice could be made optimally without observing the number of surviving children from the first period. We thus obtain the following proposition.

PROPOSITION 5. The number of births in the second period is a decreasing integer function of the number of surviving children from the first period.

Choice in the First Period

The analysis of the choice in the first period differs from that described above. Let the expected utility from a given number of births in the first period be denoted by

$$U^1(n_1, s_1, s_2) \equiv \sum_{N_1} b(N_1, n_1, s_1)V^2(N_1, s_2), \tag{33}$$

where V^2 was defined in (26). Let $n_1(s_1, s_2)$ denote the largest optimal value of n_1. Then (24) can be stated as

$$V^1(s_1, s_2) \equiv \max_{n_1} U^1(n_1, s_1, s_2) \equiv U^1(n_1(s_1, s_2), s_1, s_2). \tag{34}$$

Define the marginal expected utility from an additional birth in the first period as $U_n^1(n_1, s_1, s_2) \equiv U^1(n_1 + 1, s_1, s_2) - U^1(n_1, s_1, s_2)$. Define U_{nn}^1 accordingly. Then analogous to (27), (28), and (29), we obtain

$$U_{nn}^1(n_1, s_1, s_2) \equiv (s_1)^2 \sum_{N_1} b(N_1, n_1, s_1)[V_N^2(N_1 + 1, s_2) - V_N^2(N_1, s_2)], \tag{35}$$

where

$$V_N^2(N_1, s_2) \equiv V^2(N_1 + 1, s_2) - V^2(N_1, s_2),$$

$$\frac{\partial}{\partial s_1} U_n^1(n_1, s_1, s_2) = \frac{1}{s_1}[U_n^1(n_1, s_1, s_2) + n_1 U_{nn}^1(n_1 - 1, s_1, s_2)], \tag{36}$$

and

$$\frac{\partial}{\partial s_1} U^1(n_1, s_1, s_2) = \frac{n_1}{s_1} U_n^1(n_1 - 1, s_1, s_2). \tag{37}$$

An optimum is characterized by

$$U_n^1(n_1(s_1, s_2) - 1, s_1, s_2) \geq 0 > U_n^1(n_1(s_1, s_2), s_1, s_2). \tag{38}$$

The concavity property of the expected utility U^1, with respect to n_1, cannot be easily established at the present level of generality. Accordingly, in contrast to part i of proposition 4, the results concerning the uniqueness of the optimal n_1 cannot be easily obtained. To keep the paper brief, therefore, I shall consider only those small changes in the parameters that alter the optimal n_1 by at most one and then examine whether the optimal choice decreases or increases.

The Effect of a Change in the Mortality Rate of a First-Period Child on the First-Period Choice

The effect of a change in s_1 on $n_1(s_1, s_2)$ is described by the proposition below. The proof is omitted because it is identical to that of (15). This can be verified using (36) and (38).

PROPOSITION 6. The number of births in the first period is nondecreasing in the mortality rate of a first-period child.

The Effect of a Change in the Mortality Rate of a Second-Period Child on the First-Period Choice

To analyze this effect, we need the following two results from an envelope theorem for integer choice variables (see Sah and Zhao [1990] for these and related results).

i) Consider the optimization in (26). If the optimal value of n_2 is unique, then the derivative $\partial V^2/\partial s$ exists and the standard envelope theorem holds:

$$\frac{\partial}{\partial s_2} V^2(N_1, s_2) = \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\bigg|_{n2=n2(N1,s2)}. \tag{39}$$

The same theorem holds even if there are two optimal values of n_2, provided

$$\frac{\partial}{\partial s_2} U_n^2(N_1, n_2 - 1, s_2)\bigg|_{n2=n2(N1,s2)} = 0. \tag{40}$$

ii) In general, the standard envelope theorem may not hold and the derivative in (39) may not exist when there are two optimal values of n_2. However, the right-hand and the left-hand derivatives, denoted, respectively, by $\partial V^{2+}/\partial s$ and $\partial V^{2-}/\partial s$, always exist, and the corresponding envelope theorems are

$$\begin{aligned} \frac{\partial}{\partial s_2} V^{2+}(N_1, s_2) &= \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\bigg|_{n2=n2(N1,s2)-1}, \\ \frac{\partial}{\partial s_2} V^{2-}(N_1, s_2) &= \frac{\partial}{\partial s_2} U^2(N_1, n_2, s_2)\bigg|_{n2=n2(N1,s2)}. \end{aligned} \tag{41}$$

With these two results, the following proposition can be established (see Sah [1990] for a proof).

PROPOSITION 7. The number of births in the first period is nondecreasing in the mortality rate of a second-period child if (*a*) a unit decrease in the number of surviving children from the first period does not induce more than a unit increase in the maximum or the minimum number of optimal births in the second period, and (*b*) $u_{12}(N_1, N_2) \leq s_2 u_{22}(N_1, N_2)$.

To interpret condition *a*, recall from proposition 5 that the number of births in the second period either increases or remains unchanged if one fewer child from the first period survives. Condition *a* restricts how large this increase can be. This restriction is consistent with empirical studies, which typically show that the increase in births per death is considerably smaller than one (see Schultz 1981, chap. 5; 1988, pp. 444–45).

To interpret condition *b*, recall that $u_{22} < 0$ and $u_{12} < 0$. Thus condition *b* can be restated as

$$|u_{12}(N_1, N_2)| \geq s_2 |u_{22}(N_1, N_2)|. \qquad (42)$$

That is, the decrease in the marginal ex post utility of a surviving child from the second period, due to one more surviving child from the first period, is not significantly smaller than the corresponding decrease due to one more surviving child from the second period.

The Effect of a Change in the Mortality Rate on the Expected Number of Total Births

Ex ante, the number of births in the second period, $n_2(N_1, s_2)$, is a random variable contingent on N_1. The number of total births is thus random. The expected number of total births, denoted by $e(s_1, s_2)$, is

$$e(s_1, s_2) = n_1(s_1, s_2) + \sum_{N_1=0}^{n_1(s_1, s_2)} b(N_1, n_1(s_1, s_2), s_1) n_2(N_1, s_2). \qquad (43)$$

A question that arises, then, is, What are the effects of changes in s_1 and s_2 on $e(s_1, s_2)$? This question is not fully answered by the preceding analysis. For instance, even if the number of births in each of the two periods, $n_1(s_1, s_2)$ and $n_2(N_1, s_2)$, is nonincreasing in s_2, it does not follow that e is nonincreasing in s_2. The reason is that a smaller number of births in the first period can reduce the number of first-period children who survive. In turn, this can raise the number of births in the second period.

Let us examine those changes in s_1 and s_2 that induce local perturbations in the optimal choice of n_1 and n_2. First, consider the effect of a change in s_1 on e. From (43), s_1 affects e in two ways: (i) it directly affects the density b in the second term on the right-hand side of (43), and (ii) it affects $n_1(s_1, s_2)$. The latter effect influences the first term on the right-hand side of (43) and also the density b in the second term. Consider these two effects separately. The first effect of a larger s_1 is to induce a first-order stochastic improvement in the density b. Further, from proposition 5, $n_2(N_1, s_2)$ is nonincreasing in N_1. Therefore, using a standard result concerning first-order stochastic dominance (see Ingersoll 1987, p. 123), we obtain

$$\frac{\partial}{\partial s_1} e(s_1, s_2)\bigg|_{n1=n1(s1,s2)} \leq 0. \tag{44}$$

The second effect of s_1 on e occurs through $n_1(s_1, s_2)$. Proposition 6 shows that the optimal n_1 is nonincreasing in s_1. Now if a change in s_1 does not affect $n_1(s_1, s_2)$, then e is not influenced by the effect of s_1 under consideration at present. Therefore, let us examine the case in which $n_1(s_1', s_2) = n_1(s_1, s_2) - 1$ for a value of s_1' larger than s_1. For notational brevity, denote $n_1(s_1, s_2)$ by q. Then, using (43), I show in the Appendix that

$$\begin{aligned} e(s_1', s_2) - e(s_1, s_2) = -1 + s_1 \sum_{N_1=0}^{q-1} b(N_1, q-1, s_1) \\ \times [n_2(N_1, s_2) - n_2(N_1 + 1, s_2)]. \end{aligned} \tag{45}$$

Now assume that condition a of proposition 7 holds. Then a unit increase in N_1 induces no more than a unit decrease in $n_2(N_1, s_2)$. Correspondingly, the right-hand side of (45) is negative because $s_1 < 1$. Putting together the two effects of s_1 on e, we conclude that e is nonincreasing in s_1 if condition a holds.

Next, consider the effect of a change in s_2 on e. From (43), s_2 has two effects as well: it affects $n_2(N_1, s_2)$ and $n_1(s_1, s_2)$. The analysis of the first effect is straightforward. Part ii of proposition 4 shows that $n_2(N_1, s_2)$ is nonincreasing in s_2. Thus (43) yields

$$\frac{\partial}{\partial s_2} e(s_1, s_2)\bigg|_{n1=n1(s1,s2)} \leq 0. \tag{46}$$

To analyze the second effect, recall from proposition 7 that $n_1(s_1, s_2)$ is nonincreasing in s_2, provided that conditions a and b hold. The analysis of this effect is thus similar to the earlier analysis of the effect of s_1 on e due to the induced change in $n_1(s_1, s_2)$. Hence, because of this effect, a larger s_2 does not raise e if conditions a and b hold. Putting the two effects together, we conclude that e is nonincreasing in s_2 if conditions a and b hold.

Proposition 8 summarizes these results.[7]

PROPOSITION 8. The expected number of total births is nondecreasing in the mortality rate of either a first-period child or a second-period child if conditions *a* and *b* of proposition 7 hold.

The Effect of a Change in the Mortality Rate on Parental Welfare

The parental welfare level, described by $V^1(s_1, s_2)$ in (34), is affected by s_1 and s_2 as follows.

PROPOSITION 9. Parental welfare does not decrease if the mortality rate of either a first-period child or a second-period child decreases. The welfare strictly increases if a decrease in either mortality rate induces a nontrivial change in either the first-period or the second-period fertility choice.

Analogous to the proof of proposition 3, the effect of s_1 on V^1 can be established using (34), (37), and (38). To examine the effect of s_2, suppose for a moment that the optimal choice of n_1 is left unchanged, although the value of s_2 has increased to s_2'. Then it is clear from (33), (34), and part iii of proposition 4 that V^1 does not decrease and that V^1 strictly increases if the higher value of s_2 nontrivially alters the choice of n_2 for even one value of N_1. Compared to this outcome, the actual parental welfare will not be lower because n_1 will also be chosen optimally, given the changed value of s_2. Thus $V^1(s_1, s_2') \geq V^1(s_1, s_2)$, and the preceding inequality is strict if the change in s_2 nontrivially alters the choice of n_1 or any one of the choices of n_2 that the parents might make in the future.

Extensions

Interim Utility and Mortality

A simplification employed in the two-stage model analyzed above was that the mortality of the children born in the current period, insofar as it is a critical determinant of the current fertility decision, is revealed before the next set of decisions is made. This can be modified to incorporate age-specific mortality, as well as the interim utility (once again, net of all benefits and costs) that the parents derive in a

[7] These results can be strengthened in several ways. For instance, *e* is strictly lowered by a larger s_1 provided that the second-period choice, $n_2(N_1, s_2)$, is not entirely insensitive to N_1. The reason is that inequality (44) is strict if $n_2(N_1, s_2)$ is decreasing in N_1 for even one value of N_1. Similarly, *e* is strictly lowered by a larger s_2 provided that a larger s_2 lowers $n_2(N_1, s_2)$ for even one value of N_1. The reason is that inequality (46) is strict in this case.

particular period from the surviving children from a previous period, some of whom might die in the near future. As an illustration, let the random variable N_1 denote the number of first-period children surviving at the end of the period, when the second-period choice, n_2, is made. Let the random variable $N_{1,2}$ denote the number, out of N_1, who survive until the end of the second period. Let $s_{1,2}$ denote the corresponding probability of each such survival. Denote the utility beyond the second period by $u(N_{1,2}, N_2)$ and the interim (age-specific) utility during the second period by $v(N_1)$. Then the second-period choice continues to be represented by (25) and (26), provided that the u on the right-hand side of (25) is replaced by $\Sigma_{N1,2}\, b(N_{1,2}, N_1, s_{1,2})u(N_{1,2}, N_2)$. The first-period choice also continues to be described by (33) and (34), provided that the V^2 on the right-hand side of (33) is replaced by $v(N_1) + V^2(N_1, s_2)$. Further, the ex ante costs of births in each of the two periods can be incorporated. In this case, as the analysis presented at the end of Section II indicates, additional conditions will be needed for some of the results.

Multiple Stages of Choice

Consider, briefly, the following multistage extension of the simple two-stage choice model analyzed earlier. Let $t = 1, 2, \ldots, T$ denote the different periods of choice, where $t = 1$ denotes the first period and $t = T$ denotes the last. Let $n_t = 1$ or 0 denote whether or not a birth occurs in period t. Assume that, to the extent a child's mortality critically influences fertility decisions, it is experienced within one period after the child's birth. Such an assumption is often employed in the context of developing countries because a large portion of child mortality is experienced within the very early phase of life. In any case, this assumption can be modified to incorporate age-specific mortality, as discussed in the previous paragraph. Let the random variable N_t denote the number of children surviving out of n_t. Let s_t denote the survival probability of a child born in period t. Define the vectors $\mathbf{N}_t \equiv (N_1, \ldots, N_{t-1})$ and $\mathbf{s}_t \equiv (s_t, \ldots, s_T)$. Then the fertility choice is described by

$$V^t(\mathbf{N}_t, \mathbf{s}_t) \equiv \max_{n_t} \sum_{N_t} b(N_t, n_t, s_t)V^{t+1}(\mathbf{N}_{t+1}, \mathbf{s}_{t+1}) \quad \text{for } t = 1 \text{ to } T. \quad (47)$$

In (47), $V^{T+1} \equiv u(\mathbf{N}_{T+1})$ is the ex post utility; interim utilities can be included as discussed earlier. The parental welfare level is described by the indirect utility $V^1(\mathbf{s}_1)$. Let $n_t(\mathbf{N}_t, \mathbf{s}_t)$ denote the largest optimal value of n_t.

The foregoing problem can be analyzed using the methods developed in this section. For instance, the following results can be established: (i) The value $n_t(\mathbf{N}_t, \mathbf{s}_t)$ is nonincreasing in s_t. (ii) The parental welfare level, V^1, is nondecreasing in $(s_1, \ldots, s_T)$, and it is strictly increasing in each s_t if a larger value of s_t has a nontrivial effect on any of the fertility choices.

IV. Concluding Remarks[8]

Remarks on Some Earlier Models

Ben-Porath and Welch (1972) and Ben-Porath (1976) have examined a single-stage choice model with the following specification of the expected utility of n births:

$$U(n, s) = G(ns, M - pn), \tag{48}$$

where p denotes the ex ante cost per birth, M denotes parental income, and n is treated as a continuous variable. A motivation that they suggest for (48) is that the parents are concerned about the expected number of surviving children, ns. Variants of this model that they consider do not alter the particular aspects that are of interest in the present discussion. Let η_{ns} and η_{np} denote the elasticities of n with respect to s and p, respectively. Then (48) yields $\eta_{ns} = -(1 + \eta_{np})$. This relationship implies, in general, an ambiguous fertility effect of a change in the mortality rate (see Heckman and Willis [1976] and Schultz [1976] for discussions of this model, and Birdsall [1988, pp. 518–19] and Schultz [1981, pp. 131–34] for interpretations of the ambiguity just noted). On the other hand, the expected number of surviving children, denoted by $E \equiv ns$, is larger if s is larger. To see this, let η_{Es} and η_{Ep} denote the elasticities of E with respect to s and p, respectively. Then $\eta_{Es} = 1 + \eta_{ns} = -\eta_{np} = -\eta_{Ep} > 0$, under the reasonable assumption that E has the property of a normal good (i.e., $\eta_{Ep} < 0$).[9]

The specification in (48) is, in general, inconsistent with parental choice under uncertainty. The reason is that the expected value of a function of a random argument can be expressed as a function solely of the expected value of the random argument (and not of its higher moments) only if the original function is linear in the random argu-

[8] I have benefited from discussions with James Heckman on the material presented in this section.

[9] Barro and Becker have analyzed dynamic population models based on dynastic utility (see Becker and Barro 1988; Barro and Becker 1989). They focus on the expected number of surviving children, E, rather than on the number of births, n, which is the focus of the present paper and of other papers cited in this subsection. A part of their analysis deals with the effect of a change in s on E, using a utility function similar to that in (48) in which the utility depends on E. They use the effect just noted, namely, that E is increasing in s. This effect, however, may arise in other models as well. In our analysis, n is shown to be a decreasing integer function of s. Thus E may be locally increasing or decreasing in s depending on the value of s.

ment. Thus the version of (48) that is consistent with choice under uncertainty, but that has not been analyzed by Ben-Porath and Welch, is

$$U(n, s) = G_1 ns + G_2(M - pn), \tag{49}$$

where the parameter G_1 does not depend on n, and G_2 is a function. The specification in (49) predicts, contrary to the pattern typically observed, that fertility increases if the mortality rate declines; that is, $\eta_{ns} > 0$. It also implies that the ex post net utility from N surviving children is linear in N. From the discussion at the beginning of Section II, this assumption is problematic.

O'Hara (1975) has examined a single-stage model in which the ex post utility is $u(Z, N, Q)$, where Z denotes parental consumption and Q denotes the quality of the children. The quality of the children yields parental benefits only if the children survive beyond some stage ("maturity"). His analysis is based on the maximization of the following expected utility: $p_1 u(Z, 0, 0) + p_2 u(Z, n, 0) + p_3 u(Z, n, Q)$, subject to a standard budget constraint. The only three outcomes that are relevant in this specification are that (i) *all* n children die after birth, (ii) *all* n children survive before maturity but *none* to maturity, and (iii) *all* n children survive to maturity. The respective probabilities of these outcomes are p_1, p_2, and p_3. This model assigns no utility to all those outcomes in which some but not all of the children die. It also treats the probabilities p_1, p_2, and p_3 as exogenous parameters, whereas these probabilities depend, in general, on the number of births, n, which is a choice variable. For instance, in our notation, $(1 - s)^n$ is the probability that all n children will die. An extreme assumption that could possibly rationalize this model is that the deaths of children are completely correlated so that, regardless of the number of children, either they all live or they all die.

Remarks on the Use of a Discrete Representation

A discrete representation of the number of children, born or surviving, is obviously more realistic than a continuous representation. A discrete representation also yields crisper and better results in the present context. To see this, reconsider propositions 1, 2, and 3, using the simple single-stage model described in the beginning of Section II. Let $f(N, n, s)$ denote the probability density of N survivals out of n births, where N and n are now treated as continuous variables. The expected utility is now $U(n, s) = \int_0^n u(N) f(N, n, s) dN$ instead of (2). Let a subscript denote the variable with respect to which a partial derivative is being taken. Assume that the optimal value of n, denoted by $n(s)$, is interior. Then instead of (9a) and (9b), an optimality condition now is $U_n(n, s) = 0$ at $n = n(s)$. To establish the continuous versions of propositions 1, 2, and 3, one would need to show, respectively, the following:

$$U_{nn}(n, s) < 0 \quad \text{for all } n, \tag{50}$$

$$\frac{U_{ns}(n, s)}{U_{nn}(n, s)} \geq 0 \quad \text{at } n = n(s), \tag{51}$$

$$U_s(n, s) \geq 0 \quad \text{at } n = n(s). \tag{52}$$

These expressions are examined in the Appendix, where it is shown that they do not follow from a set of assumptions that are either intuitive or comparable to those employed in our discrete analysis.

One reason for this difference between a discrete and a continuous representation is as follows. To analyze the problem at hand, we need to evaluate the induced changes in the probabilities of various numbers of survivals (and the induced changes in expressions containing these probabilities) when n and s change. In the discrete case, these induced changes need to be evaluated only at integer values of N. In the continuous case, these changes need to be evaluated on the entire real line representing N. Moreover, the evaluation of these induced changes is greatly simplified in the discrete case when the survival probabilities are described by a functional form such as the binomial density. The reason is that the binomial density has highly tractable properties (arising partly because of the independence of discrete outcomes) that are lost to a degree even when a comparable continuous density (e.g., a normal approximation of the binomial density) is used. Finally, it is apparent from the analysis in this paper that a discrete representation can be helpful in other, more complex, models of fertility choice. Thus, in the present context, tractability and realism go hand in hand.

Appendix

Derivation of (4) and (5)

The relationship described in (A1) is used repeatedly below. This "partial summation formula" (see Rudin 1976, p. 70) is the discrete equivalent of integration by parts. Let x_i's and y_i's denote any set of numbers. Define $X_i \equiv \Sigma_{j=0}^{i} x_j$. Then

$$\sum_{i=0}^{n} x_i y_i = -\sum_{i=0}^{n-1} X_i(y_{i+1} - y_i) + X_n y_n. \tag{A1}$$

Also, denote the cumulative binomial density by $B(N, n, s) \equiv \Sigma_{j=0}^{N} b(j, n, s)$.

To derive (4), note that (2), (A1), and the definition $B(n, n, s) = 1$ yield

$$U(n, s) = -\sum_{N=0}^{n-1} B(N, n, s) u_N(N) + u(n). \tag{A2}$$

In turn,

$$U_n(n, s) = -\sum_{N=0}^{n-1} [B(N, n+1, s) - B(N, n, s)]u_N(N) + [1 - B(n, n+1, s)]u_N(n). \tag{A3}$$

To evaluate (A3), we need two identities. First,

$$B(N, n+1, s) - B(N, n, s) = -sb(N, n, s) \tag{A4}$$

(see Sah [1989] for a general version of this identity). Second, by definition, $1 - B(n, n+1, s) = s^{n+1} = sb(n, n, s)$. From these, (A3) can be expressed as

$$U_n(n, s) \equiv U(n+1, s) - U(n, s) = s\sum_{N=0}^{n} b(N, n, s)u_N(N), \tag{A5}$$

which is (4). Next, we use (A5) again but substitute U_n in the place of U. This yields (5).

Derivation of (6) and (7)

A property of the binomial cumulative density is

$$\frac{\partial}{\partial s}B(N, n, s) = -nb(N, n-1, s). \tag{A6}$$

A version of (A6) appears as equation (10.9) in Feller (1968, p. 173). Further, (A1) and (A5) yield

$$U_n(n, s) = s\left[-\sum_{N=0}^{n-1} B(N, n, s)u_{NN}(N) + u_N(n)\right]. \tag{A7}$$

From (A6) and (5), the derivative of (A7), with respect to s, can be rearranged to yield (6). Next, from (A2) and (A6), $\partial U(n)/\partial s = n\,\Sigma_{N=0}^{n-1}\, b(N, n-1, s)u_N(N)$. This expression and (A5) yield (7).

Derivation of (22) and Evaluation of (23)

Write $\phi(N, n-1, s)$ as $\phi(N)$. Using (A7) and (5) and recalling that $\partial U_n'(n, s)/\partial s$ is given by the right-hand side of (6), we obtain (22), where $\phi(N) = nsb(N, n-1, s) - B(N, n, s)$. In turn, from (A4), the definition of ϕ in (23) follows.

We now establish that

$$\phi(N) > 0 \quad \text{for } N = 0 \text{ to } n-1 \text{ if } s > .81 \text{ and } n \leq 12. \tag{A8}$$

First, it can be shown that $\phi(N)$ is single-peaked in N (see Sah 1990). Thus $\phi(N)$ achieves a minimum at $N = 0$ or $N = n - 1$. Consequently, $\phi(N) > 0$ for $N = 0$ to $n - 1$ if $\phi(0) > 0$ and $\phi(n-1) > 0$. From (23), $\phi(0) = [(n+1)s - 1](1-s)^{n-1}$. Since $n \geq 1$, it follows that $\phi(0) > 0$ if $s > .5$. Next, from (23), $\phi(n-1) = (n+1)s^n - 1$, which is positive if $s > \gamma(n)$, where $\gamma(n) \equiv \exp\{-[\ln(n+1)]/n\}$. Further, $\gamma(n) > \gamma(n-1)$. Hence, if $s > \gamma(n^*)$, then $s > \gamma(n)$ for $n \leq n^*$. Since $\gamma(12) = .81$, it follows that $\phi(n-1) > 0$ if $s > .81$ and $n \leq 12$. Thus (A8) is established.

Derivation of (31)

Analogous to (A5), we now have

$$U_n^2(N_1, n_2, s_2) = s_2 \sum_{N_2=0}^{n_2} b(N_2, n_2, s_2)u_2(N_1, N_2).$$

The definition of $u_{12}(N_1, N_2)$ and its negative sign then yield (31) because

$$U_n^2(N_1 + 1, n_2, s_2) - U_n^2(N_1, n_2, s_2) = s_2 \sum_{N_2=0}^{n_2} b(N_2, n_2, s_2)u_{12}(N_1, N_2) < 0. \quad \text{(A9)}$$

Derivation of (45)

Define $Y(q) \equiv \Sigma_{N_1=0}^{q} b(N_1, q, s_1)n_2(N_1, s_2)$. Thus, from (43), $e(s_1', s_2) - e(s_1, s_2) = -1 + Y(q - 1) - Y(q)$. Next, using (A5), but substituting $Y(q)$ in the place of $U(n, s)$, we obtain

$$Y(q) - Y(q - 1) = s_1 \sum_{N_1=0}^{q-1} b(N_1, q - 1, s_1)[n_2(N_1 + 1, s_1) - n_2(N_1, s_1)].$$

Expression (45) follows.

Examination of (50), (51), and (52)

Let $F(N, n, s) \equiv \int_0^n f(j, n, s)dj$ denote the cumulative density of N survivals. Define $G(N, n, s) \equiv \int_0^N F(K, n, s)dK$. Thus $G_N(n, n, s) = 1$ and $G_{Nn}(n, n, s) = G_{NN}(n, n, s) = 0$. Using these expressions and integration by parts, we obtain

$$U_{nn}(n, s) = \int_0^n G_{nn}(N, n, s)u_{NN}(N)dN - G_{nn}(n, n, s)u_N(n). \quad \text{(A10)}$$

Now, to establish (50), we need to show that (A10) is negative. Note that the $u_{NN}(N)$ are negative and that $u_N(n)$ can be positive or negative. Thus (A10) will be negative, in general, only if

$$G_{nn}(n, n, s) = 0, \quad G_{nn}(N, n, s) \geq 0 \quad \text{for } n > N \geq 0. \quad \text{(A11)}$$

Clearly, (A11) does not represent any intuitive property of the survival probabilities. It does not describe a property of stochastic dominance, of any order, for the survival probabilities. Next, an assumption concerning the continuous density $f(N, n, s)$ that is comparable to the discrete binomial density, (1), is the one in which the latter is approximated by a normal density. For example, $f(N, n, s) = z\{(N - ns)/[ns(1 - s)]^{1/2}\}$, where z is the unit normal density. It can be easily ascertained that this or other similar approximations do not yield (A11). Similar difficulties arise in establishing the expressions in (51) and (52).

References

Barro, Robert J., and Becker, Gary S. "Fertility Choice in a Model of Economic Growth." *Econometrica* 57 (March 1989): 481–501.

Becker, Gary S. "An Economic Analysis of Fertility." In *Demographic and*

Economic Change in Developed Countries. Universities–National Bureau Conference Series, no. 10. Princeton, N.J.: Princeton Univ. Press (for NBER), 1960.

Becker, Gary S., and Barro, Robert J. "A Reformulation of the Economic Theory of Fertility." *Q.J.E.* 103 (February 1988): 1–25.

Ben-Porath, Yoram. "Fertility Response to Child Mortality: Micro Data from Israel." *J.P.E.* 84, no. 4, pt. 2 (August 1976): S163–S178.

Ben-Porath, Yoram, and Welch, Finis. "Chance, Child Traits, and Choice of Family Size." Report no. R-1117-NIH/RF. Santa Monica, Calif.: Rand Corp., 1972.

Birdsall, Nancy. "Economic Approaches to Population Growth." In *Handbook of Development Economics,* vol. 1, edited by Hollis Chenery and T. N. Srinivasan. Amsterdam: North-Holland, 1988.

Dyson, Tim, and Murphy, Mike. "The Onset of Fertility Transition." *Population and Development Rev.* 11 (September 1985): 399–440.

Feller, William. *An Introduction to Probability Theory and Its Applications.* Vol. 1. 3d ed. New York: Wiley, 1968.

Freedman, Ronald. *The Sociology of Human Fertility.* New York: Irvington, 1975.

Heckman, James J., and Willis, Robert J. "Estimation of a Stochastic Model of Reproduction: An Econometric Approach." In *Household Production and Consumption,* edited by Nestor E. Terleckyj. New York: Columbia Univ. Press (for NBER), 1976.

Ingersoll, Jonathan E., Jr. *Theory of Financial Decision Making.* Totowa, N.J.: Rowman and Littlefield, 1987.

Newman, John L. "A Stochastic Dynamic Model of Fertility." In *Research in Population Economics,* vol. 6, edited by T. Paul Schultz. Greenwich, Conn.: JAI, 1988.

O'Hara, Donald J. "Microeconomic Aspects of the Demographic Transition." *J.P.E.* 83 (December 1975): 1203–16.

Preston, Samuel H., ed. *The Effects of Infant and Child Mortality on Fertility.* New York: Academic Press, 1978.

Rudin, Walter. *Principles of Mathematical Analysis.* 3d ed. New York: McGraw-Hill, 1976.

Sah, Raaj K. "Comparative Properties of Sums of Independent Binomials with Different Parameters." *Econ. Letters* 31 (November 1989): 27–30.

———. "The Effects of Mortality Changes on Fertility Choice and Individual Welfare: Some Theoretical Predictions." Discussion Paper no. 599. New Haven, Conn.: Yale Univ., Econ. Growth Center, 1990.

Sah, Raaj K., and Zhao, Jingang. "Some Envelope Theorems for Integer and Discrete Choice Variables." Discussion Paper no. 598. New Haven, Conn.: Yale Univ., Econ. Growth Center, 1990.

Schultz, T. Paul. "Determinants of Fertility: A Micro-economic Model of Choice." In *Economic Factors in Population Growth,* edited by Ansley J. Coale. New York: Wiley, 1976.

———. *Economics of Population.* Reading, Mass.: Addison-Wesley, 1981.

———. "Economic Demography and Development." In *The State of Development Economics: Progress and Perspectives,* edited by Gustav Ranis and T. Paul Schultz. Oxford: Blackwell, 1988.

Wolpin, Kenneth I. "An Estimable Dynamic Stochastic Model of Fertility and Child Mortality." *J.P.E.* 92 (October 1984): 852–74.