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AN EXPLICIT CLOSED-FORM FORMULA FOR PROFIT-MAXIMIZING k-OUT-OF-n SYSTEMS SUBJECT TO TWO KINDS OF FAILURES

RAAJ K. SAH

Department of Economics, University of Chicago,
1126 East 59th Street, Chicago,
IL 60637, U.S.A.

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Abstract: This paper derives and analyzes an explicit closed-form formula for the optimal k in k-out-of-n systems consisting of i.i.d. components. The system can be in one of two possible modes with a pre-specified probability. The components are subject to failure in each of the two modes. The costs of the two kinds of system failures are generally not identical. Since the formula is explicit, it permits a calculation of the optimal k directly in terms of the parameters of the system. In addition, it yields many results concerning both the bounds of the optimal k and the effects of a change in parameters on the optimal k and on the optimized value of the system's expected profit.

I. INTRODUCTION

This paper studies the design of optimal systems using unreliable components. The system under consideration consists of n identical and statistically independent components. The system can be, with a pre-specified probability, in one of two possible modes: mode 1, in which the components are commanded to close; or mode 2, in which the components are commanded to open. A component is subject to failure in each mode: in mode 1 it may fail to close, and in mode 2 it may fail to open. The system is closed if k or more components are closed; otherwise it is open. Thus, the two types of potential failures of the system are: failure to close (which occurs if fewer than k components close when the system is in mode 1), and failure to open (which occurs if k or more components close when the system is in mode 2). These two kinds of system failures can have different costs. Our objective, then, is to study the optimal k, referred to as k^*, treating other features of the system as parameters. The criterion for choosing k^* is the maximization of the system's expected profit.

The contribution of this paper is as follows. We derive and analyze an *explicit closed-form formula* for k^*. Using this formula, k^* can be calculated directly in terms of the parameters. In addition, this formula yields a number of results concerning the properties of k^*; for example, we determine the bounds of k^*, and the direction and magnitude of change in k^* due to a change in parameters. We also present some results on the impact of a change in parameters on the optimized value of the system's expected profit. All of these results are exact; they do not require any approximations.

A brief background to the problem studied in this paper is as follows. In a recent paper, Sah and Stiglitz (1988a) presented an *implicit* characterization of k^* for a similar system. (Since this characterization

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is implicit, it does not permit a direct calculation of k^* in terms of the parameters, as the formula reported in the present paper does.) They analyzed k^* using this implicit characterization and the following two approximations: (i) the derivatives of the binomial probability density are approximated by the derivatives of the normal probability density, and (ii) k, n, and k^* are treated as continuous rather than integer variables. This approach did not permit them to obtain most of the results (concerning the bounds of k^* and the effects of a change in parameters on k^*) reported in the present paper, while the results that they did obtain were subject to the approximations just noted. Another set of effects studied in the present paper (namely, the effects of a change in parameters on the optimized value of the system's expected profit) is not examined in Sah and Stiglitz, nor, to our knowledge, has it been elsewhere in the literature.

A special case of the problem studied here is one in which it is assumed that: (i) the costs of the two kinds of system failures are identical, and (ii) the system is in the two modes with equal probability. In this case, the maximization of the system's expected profit is the same as the maximization of the system's reliability, where the latter is defined as the probability of the system's success in mode 1 minus the probability of the system's failure in mode 2. This special case has been analyzed by Ben-Dov (1980), and its variants have been examined by Ansell and Bendell (1982), and Phillips (1980). These authors also provide earlier citations.

Systems of the type studied in the present paper are of practical importance in engineering contexts such as relay circuits and monitoring safety systems (see Barlow and Proschan (1981), Ben-Dov (1980) and references therein). The analysis of such systems is also useful in studying the performance and design of human organizations such as committees and hierarchies (see Sah and Stiglitz (1988b)). For example, consider a committee with n members that accepts a project (or an idea) if k or more members accept it. If there are two types of projects (good and bad) and if each member's judgment is fallible concerning both types of projects, then some aspects of this committee's performance can be modeled along the lines of the system studied in this paper.

The formula for k^* is derived in Section II. Section III presents the bounds of k^*. Section IV described the results concerning the effects of a change in parameters on k^*. Section V analysis the effects of a change in parameters on the optimized value of the system's expected profit.

II. THE FORMULA FOR THE OPTIMAL k

Let q_1 denote the probability of a component's failure when the system is in mode 1; that is, failure to close. Let q_2 denote the probability of a component's failure when the system is in mode 2; that is, failure to open. Assume that $1 > q_i > 0$, for $i = 1$ and 2. Define $b(j, n, q_i) = \binom{n}{j} q_i^j (1 - q_i)^{n-j}$ to be the density of a binomial variate with parameters (n, q_i). Define the corresponding cumulative density $B(k, n, q_i) = \sum_{j=0}^{k} b(j, n, q_i)$. Recalling the verbal definition of the system, then, the system's probability of failure in mode 1 is $B(k-1, n, 1-q_1)$. In mode 2, the system's probability of failure is $1 - B(k-1, n, q_2)$. It might be noted here that if one were to use the terminology employed by the *IEEE Transactions on Reliability*, then our system would be k-out-of-n:G in mode 1, and k-out-of-n:F in mode 2.

Let α denote the probability the system will be in mode 1; assume $1 > \alpha > 0$. Let π^1 and π^2 respectively denote the gain from the system's success and failure in mode 1. For mode 2, the corresponding gains are denoted by π^3 and π^4. A negative value of a π^j signifies a loss. We assume that $\pi^1 > \pi^2$ and $\pi^3 > \pi^4$.

The expected profit of the system is

$$\Pi = \alpha[\pi^1\{1 - B(k-1, n, 1-q_1)\} + \pi^2 B(k-1, n, 1-q_1)] + (1-\alpha)[\pi^3 B(k-1, n, q_2) + \pi^4\{1 - B(k-1, n, q_2)\}] . \quad (1)$$

Maximizing this expected profit with respect to k is the same as maximizing

$$Y(k) = -B(k-1, n, 1-q_1) + \beta B(k-1, n, q_2) , \quad (2)$$

where we have defined a summary parameter $\beta = (1-\alpha)(\pi^3 - \pi^4)/\alpha(\pi^1 - \pi^2)$. From above, $\beta > 0$. The effects of the parameters $\{\alpha, \pi^1, \pi^2, \pi^3, \pi^4\}$ on β are easily ascertained: $\partial\beta/\partial\alpha < 0$, $\partial\beta/\partial\pi^1 < 0$, $\partial\beta/\partial\pi^2 > 0$, $\partial\beta/\partial\pi^3 > 0$, and $\partial\beta/\partial\pi^4 < 0$. The feasible values of k run from 0 to n.

A special case of the above formulation is one in which it is assumed that the gain from the system's success in either mode is zero (i.e., $\pi^1 = \pi^3 = 0$), the gains from the system's failure is the same in the two modes (i.e., $\pi^2 = \pi^4$), and the system is in the two modes with equal probability (i.e., $\alpha = 1/2$). Since $\beta = 1$ in this special case, the maximization of (2) is the same as maximizing the system's reliability, defined as $\{1 - B(k-1, n, 1-q_1)\} - \{1 - B(k-1, n, q_2)\}$. As was noted earlier, this case has been examined in the literature. The results corresponding to this special case can be easily identified in the more general analysis below.

Unless stated otherwise, we shall assume throughout that $1 - q_1 > q_2$. (Systems that do not satisfy this condition are discussed at the end of this section.) Using this assumption, it is shown in the Appendix that:

The optimal value of k is either unique, or there are two neighboring values of k that are both optimal. (3)

If the optimal value of k is unique, we denote it as k^*. If two values of k are optimal, we denote them as k^* and $k^* + 1$. Now, consider those cases in which k^* is interior; that is, $n - 1 \geq k^* \geq 1$. Given (3), k^* must satisfy:

$$Y(k^*) - Y(k^* + 1) \geq 0 , \text{ and } Y(k^*) - Y(k^* - 1) > 0 . \quad (4)$$

For notational brevity, define $t = (1-q_1)/q_2$ and $r = q_1/(1-q_2)$. Also, define

$$K = \frac{\ell n\, \beta - n \ell n\, r}{\ell n(t/r)} . \quad (5)$$

Note that $t/r > 1$, because $t > 1$ and $r < 1$. Then, by substituting (2) and the definition of B into (4), the expressions in (4) can be restated as

$$k^* \geq K , \text{ and } K > k^* - 1 . \quad (6)$$

Define $[K]_+$ to be the smallest integer equal to or larger than K. Then, (4) and (6) yield

THEOREM 1

$k^* = [K]_+$, *where* K *is given by* (5). (7)

This closed-form formula permits a simple calculation of the optimal k directly in terms of the parameters. Also, it is easily verified from (4), (6), and (7) that: (i) if K is not an integer, then the optimal value of k is unique, and (ii) if K is an integer, then the optimal values of k are k^* and $k^* + 1$, where $k^* = K$. Moreover, necessary and sufficient conditions for a corner value of k to be optimal can also be derived from (2), the definition of B, and expressions (A3) and (A4) presented in the Appendix. These conditions are: (i) $k = 0$ is optimal if and only if $Y(0) \geq Y(1)$, or equivalently, if and only if $\beta \leq r^n$; and (ii) $k = n$ is optimal if and only if $Y(n) \geq Y(n-1)$, or equivalently, if and only if $\beta \geq rt^{n-1}$.

The analysis below uses the following inequalities, all of which follow immediately from the definitions of the terms involved.

$$\ell n\, t > 0, \quad \ell n\, r < 0, \quad \text{and} \quad \ell n(t/r) > 0. \tag{8}$$

$$\ell n\, tr \gtrless 0 \text{ if } q_1 \gtrless q_2. \quad \ell n\, \beta \gtrless 0 \text{ if } \beta \gtrless 1. \tag{9}$$

For later use, it is established in the Appendix that

$$1 - q_1 > -\frac{\ell n\, r}{\ell n(t/r)} > q_2. \tag{10}$$

Also for later use, the following is obtained from (8) and (9):

$$-\frac{\ell n\, r}{\ell n(t/r)} - \frac{1}{2} = -\frac{1}{2}\frac{\ell n\, tr}{\ell n(t/r)} \gtrless 0 \text{ if } q_1 \lessgtr q_2. \tag{11}$$

Finally, note that the assumption $1 - q_1 > q_2$ may appear arbitrary, but it has often been the only case treated in the literature, thus neglecting the analysis of systems which do not satisfy this assumption (see, for example, Ben-Dov (1980) and Sah and Stiglitz (1968a)). A complete analysis is as follows. First, consider the case in which $1 - q_1 = q_2$. Then from (2), $Y(k) = (\beta - 1)B(k-1, n, q_2)$. Since B is strictly increasing in k, it follows that: (i) $k = 0$ is optimal if $\beta < 1$, (ii) $k = n$ is optimal if $\beta > 1$, and (iii) any value of k is optimal if $\beta = 1$. Next, consider the case in which $1 - q_1 < q_2$. We show in the Appendix that:

If $1 - q_1 < q_2$, then only the two polar values of k can be optimal.
$k = 0$ is optimal if $\beta < \{1 - (1-q_1)^n\}/(1 - q_2^n)$. $k = n$ is optimal otherwise. (12)

III. BOUNDS OF THE OPTIMAL k

Expressions (5), (6), (8) and (10) yield

THEOREM 2

$$\text{(i) } k^* > nq_2 \text{ if } \beta \geq 1. \tag{13}$$

$$\text{(ii) } k^* < n(1 - q_1) + 1 \text{ if } \beta \leq 1. \tag{14}$$

This theorem establishes bounds on the value of k^*, conditioned solely upon the value of β. A different set of bounds on k^*, conditioned upon the value of β as well as on the relative values of q_1 and q_2 is obtained from (5), (6), (8) and (11):

(i) $k^* > \frac{n}{2}$ if $\beta > 1$ and $q_1 \leq q_2$. (ii) $k^* < \frac{n}{2} + 1$ if $\beta < 1$ and $q_1 \geq q_2$.
(iii) $k^* = \frac{n+1}{2}$ for odd n, and $k^* = \frac{n}{2}$ or $\frac{n}{2} + 1$ for even n, if $\beta = 1$ and $q_1 = q_2$. (15)

IV. THE EFFECTS OF A CHANGE IN PARAMETERS ON THE OPTIMAL k

The closed-form formula for k^* given in (5) and (7) permits a comprehensive assessment of how k^* changes if the parameters $\{n, \beta, q_1, q_2\}$ change. Below, we assess the effects of a change in these parameters on K. The corresponding effects on k^* are obtained by a simple reinterpretation. For instance, let θ denote a parameter and let the function $K(\theta)$ denote the corresponding value of K. If we show that the change in $K(\theta)$ due to a change in θ is positive (negative), then it follows that this change in θ does not decrease (increase) k^*. It is assumed below that K is interior.

Theorem 3 presents the effects of a change in n on K. Theorem 4 presents the effects of changes in q_1 and q_2. The proofs of these theorems are given in the Appendix. The effect of a change in β on K is straightforward to assess. From (5) and (8), $\partial K/\partial\beta > 0$.

Note that, in Theorem 3, $\Delta K(n) = K(n+1) - K(n)$ denotes the change in K due to a unit change in n, whereas $\Delta\left(\frac{K(n)}{n}\right) = \frac{K(n+1)}{n+1} - \frac{K(n)}{n}$ denotes the change in the ratio K/n due to a unit change in n.

THEOREM 3

$$\text{(i)}\quad 1 - q_1 > \Delta K(n) > q_2 . \tag{16}$$

$$\text{(ii)}\quad \Delta K(n) \gtrless \frac{1}{2}, \text{ if } q_1 \lessgtr q_2 . \tag{17}$$

$$\text{(iii)}\quad \Delta\left(\frac{K(n)}{n}\right) \gtrless 0, \text{ if } \beta \lessgtr 1 . \tag{18}$$

Expression (16) provides an unconditional bound on the value of ΔK. Expression (17) shows that whether ΔK is larger or smaller than one-half depends on whether q_1 is smaller or larger than q_2. Expression (18) shows that the ratio K/n is increasing or decreasing in n depending on whether β is smaller or larger than one.

THEOREM 4

$$\text{(i)}\quad \frac{\partial K}{\partial q_1} < 0, \text{ if } \beta \le 1 . \tag{19}$$

$$\text{(ii)}\quad \frac{\partial K}{\partial q_2} > 0, \text{ if } \beta \ge 1 . \tag{20}$$

$$\text{(iii)}\quad \frac{\partial K}{\partial q} \gtrless 0, \text{ if } \beta \gtrless 1, \text{ where } q = q_1 = q_2 . \tag{21}$$

Expressions (19) and (20) show how q_1 and q_2 affect K, within certain ranges of β. These results do not depend on the values of q_1 and q_2. Expression (21) deals with the special case in which a component has the same probability of failure in the two modes; that is, $q_1 = q_2$. In this case, a higher probability of component failure raises or lowers K depending on whether β is larger or smaller than one.

It might be useful to contrast this analysis briefly with that of Sah and Stiglitz (1988a). Their method was to treat k, n and k^* as continuous variables, and replace (4) by its continuous counterpart, in which k^* is characterized by $\frac{\partial Y(k^*)}{\partial k} = 0$. A perturbation of this equality with respect to a parameter θ yields $\frac{dk^*}{d\theta} = -\frac{\partial^2 Y(k,\theta)}{\partial k \partial\theta} \Big/ \frac{\partial^2 Y(k,\theta)}{\partial k^2}$, where the right-hand side is evaluated at k^*. Their evaluation of the preceding expression was carried out by approximating the derivatives (with respect to k and the parameters) of the binomial density b by the corresponding derivatives of a normal density. With this method, they derived (15), (17), (21), but not (7), (12), (13), (14), (16), (18), (19), and (20), nor the results presented below.

V. THE EFFECTS OF A CHANGE IN PARAMETERS ON THE OPTIMIZED VALUE OF THE SYSTEM'S EXPECTED PROFIT

The method we employ to evaluate these effects is as follows. If θ denotes a parameter, then let the function $k^*(\theta)$ represent the optimal value of k. For a given θ, the optimized value of the system's expected profit is represented as $G(\theta) = \Pi(k^*(\theta), \theta)$, where the function Π is described by the right-hand side of (1). Now, suppose that the value of the parameter is changed from θ to θ'. Then, the definition of the optimum implies that $G(\theta') = \Pi(k^*(\theta'), \theta') \geq \Pi(k^*(\theta), \theta')$. Recalling that $G(\theta) = \Pi(k^*(\theta), \theta)$, it follows that

$$G(\theta') > G(\theta) \text{ if } \Pi(k^*(\theta), \theta') > \Pi(k^*(\theta), \theta), \text{ and}$$

$$G(\theta') \geq G(\theta) \text{ if } \Pi(k^*(\theta), \theta') = \Pi(k^*(\theta), \theta). \tag{22}$$

We also employ the following results:

$$\frac{\partial}{\partial q_i} B(k, n, q_i) = -nb(k, n-1, q_i). \tag{23}$$

$$B(k, n, q_i) - B(k, n-1, q_i) = -q_i b(k, n-1, q_i). \tag{24}$$

A convenient source for these results is Feller (1968, p. 173). (Expressions (23) and (24), respectively, follow directly from expressions (10.9) and (10.7) in this book.)

One would expect G to be higher if either of the probabilities of a component's failure, q_1 or q_2, is lower. To confirm this, note that, from (1) and (23), $\partial\Pi/\partial q_i < 0$ for $i = 1$ and 2. Thus, from (22), $G(q_i)$ is higher if q_i is lower. It can similarly be shown that G is higher if any one of the system gains (represented by π^1, π^2, π^3 and π^4) is higher.

Next, consider the effect of a change in α (which, it will be recalled, is the probability that the system will be in mode 1). Assume that $\pi^3 = \pi^1$ and $\pi^4 = \pi^2$; that is, the gain from system success in the two modes is identical, and the gain from system failure in the two modes is identical. Then, (1) yields $\partial\Pi/\partial\alpha = (\pi^1 - \pi^2)[1 - B(k-1, n, 1-q_1) - B(k-1, n, q_2)]$. In turn, using $1 - q_1 > q_2$, $\pi^1 > \pi^2$ and (23), we obtain: $\partial\Pi/\partial\alpha > 0$ if

$$B(k-1, n, q_2) \leq 1/2. \tag{25}$$

We can now ascertain the range of k for which (25) is satisfied. If $k^*(\alpha)$ falls within this range, then, from (22), it follows that an increase in α raises G. Assuming that $n \geq 2$, it is shown in the Appendix that sufficient conditions for (25) are

(i) $k \leq q_2(n+1)$ if $q_2 \leq 1/2$; and

(ii) $k \leq (n+1)/2$ if $q_2 \geq 1/2$. (26)

Thus, for instance, if $q_2 \geq 1/2$ and $k^*(\alpha) \leq (n+1)/2$, then $\partial G(\alpha)/\partial\alpha > 0$.

Finally, consider a change in n. It is shown in the Appendix that

$$\Pi(k, n-1) \gtrless \Pi(k, n) \text{ if } Y(k+1) \gtrless Y(k). \tag{27}$$

Now, consider the case in which there are two optimal values of k, denoted by $k^*(n)$ and $k^*(n)+1$, at the current value of n. Then, from (4), $Y(k^*(n)) = Y(k^*(n)+1)$. In turn, (27) yields $\Pi(k^*(n), n-1) = \Pi(k^*(n), n)$. Therefore, from (22), $G(n-1) \geq G(n)$. That is, G cannot decrease if n is lowered.

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APPENDIX

Derivation of Expression (3). Define $\phi(k) = \{q_1/(1-q_2)\}^n\{(1-q_1)(1-q_2)/q_1q_2\}^{k-1}$. Since $1-q_1 > q_2$, it follows that $(1-q_1)(1-q_2)/q_1q_2 > 1$. Thus,

$$\phi(k+1) > \phi(k). \tag{A1}$$

Next, by substituting the definition of B into (2), it can be shown that

$$Y(k) - Y(k-1) \gtrless 0 \text{ if and only if } \beta \gtrless \phi(k). \tag{A2}$$

We now show that:

$$Y(k) - Y(k-1) > 0 \text{ if } Y(k+1) - Y(k) \geq 0. \tag{A3}$$

$$Y(k+1) - Y(k) < 0 \text{ if } Y(k) - Y(k-1) \leq 0. \tag{A4}$$

To prove (A3), note from (A2) that $Y(k+1) - Y(k) \geq 0$ implies that $\beta \geq \phi(k+1)$. In turn, using (A1), $\beta > \phi(k)$. Given (A2), this implies (A3). The proof of (A4) is analogous.

Let k^* denote an optimal value of k. That is, $Y(k^*) \geq Y(k)$ for $k = 0$ to n. Since $Y(k^*) \geq Y(k^*-1)$, it follows from (A3) that $Y(k^*) > Y(k)$ if $k < k^*-1$. Similarly, since $Y(k^*) \geq Y(k^*+1)$, it follows from (A4) that $Y(k^*) > Y(k)$ if $k > k^*+1$. Thus, a value of k smaller than k^*-1 or larger than k^*+1 cannot be optimal. Now, suppose k^*+1 is also an optimal value of k; that is, $Y(k^*) = Y(k^*+1)$. Then, k^*-1 cannot be an optimal value of k because, from (A3), $Y(k^*) > Y(k^*-1)$. This completes the derivation of (3).

Derivation of Expression (10). From the definitions of the terms involved,

$$\frac{\ell n\, r}{\ell n(t/r)} + (1-q_1) = \frac{c_1}{\ell n(t/r)}, \text{ and } \frac{\ell n\, r}{\ell n(t/r)} + q_2 = \frac{c_2}{\ell n(t/r)}, \tag{A5}$$

where $c_1 = (1-q_1)\ell n\, t + q_1\, \ell n\, r$, and $c_2 = q_2\, \ell n\, t + (1-q_2)\ell n\, r$. Define a random variable z having value $1/t$ with probability $(1-q_1)$, and value $1/r$ with probability q_1. If E is the expectation operator, then $E(z) = 1$, $\ell n\, E(z) = 0$, and $E(\ell n\, z) = -c_1$. Since $\ell n\, z$ is strictly concave in z, Jensen's inequality (see Feller (1966, p. 151)) implies: $\ell n\, E(z) > E(\ell n\, z)$. Thus, $c_1 > 0$. This result, along with (8) and the first part of (A5), yields the first half of inequality (10). The second half of (10) is proved analogously, by defining a random variable z' having value t with probability q_2, and value r with probability $(1-q_2)$.

Derivation of Expression (12). Note that (A2) continues to hold in the present case, but since $1-q_1 < q_2$, we have

$$\phi(k) > \phi(k+1), \tag{A6}$$

instead of (A1). Now, suppose for a moment that an interior value of k is optimal. That is,

$$Y(k^*) \geq Y(k) \text{ for } k = 0 \text{ to } n, \text{ where } n-1 \geq k^* \geq 1. \tag{A7}$$

From (A7), $Y(k^*) \geq Y(k^*-1)$. (A2) thus yields $\beta \geq \phi(k^*)$. In turn, from (A6), $\beta > \phi(k)$ if $k > k^*$. Thus, using (A2) we can show that $k = n$ is optimal, which contradicts (A7). Analogously, it can be shown that (A7) implies that $k = 0$ is optimal, which, in turn, contradicts (A7). Thus, $k^* = 0$ or n. Further, $k^* = n$ if $Y(n) > Y(0)$, and $k^* = 0$ otherwise. Now, $Y(0) = 0$ because, by definition, $B(k-1, n, q_i) = 0$ if $k \leq 0$. Thus, (12) follows by substituting the definition of B into $Y(n)$.

Proof of Theorem 3. (16) follows from (5) and (10). (17) follows from (5) and (11). To obtain (18), note from (5) that $\Delta\left(\frac{K(n)}{n}\right) = -\ell n\, \beta/n(n+1)\ell n(t/r)$. Then, using (8) and (9), (18) follows.

Proof of Theorem 4. For notational brevity, define $e_i = q_i(1-q_i)\ell n(t/r)$. Then, (5) yields

$$\frac{\partial K}{\partial q_1} = \{K - n(1-q_1)\}/e_1 \text{ and } \frac{\partial K}{\partial q_2} = (K - nq_2)/e_2. \tag{A8}$$

Next, note that, from (5) and (10), $K < n(1-q_1)$ if $\beta \leq 1$, and $K > nq_2$ if $\beta \geq 1$. Thus, (19) and (20) follow from (A8). To obtain (21), note that if $q = q_1 = q_2$, then $e_1 = e_2$, and $\frac{\partial K}{\partial q} = \frac{\partial K}{\partial q_1} + \frac{\partial K}{\partial q_2}$. Thus, from (A8): $\frac{\partial K}{\partial q} = (2K - n)/e_1$. Further, (5) and (11) imply that $K \gtrless n/2$ if $\beta \gtrless 1$. Thus, (21) follows.

Derivation of Expression (26). For $n \geq 2$, a result noted in Johnson and Kotz (1969, p. 53) is: $B(k, n, (k+1)/(n+1)) \leq 1/2$, if $(n-1)/2 \geq k \geq 0$. Thus, $B(k-1, n, k/(n+1)) \leq 1/2$ if $(n+1)/2 \geq k \geq 1$. Now, from (23), $B(k-1, n, q_2)$ is decreasing in q_2. Also by definition, $B(k-1, n, q_2) = 0$ if $k = 0$. Thus, it follows that: $B(k-1, n, q_2) \leq 1/2$ if $q_2 \geq k/(n+1)$ and if $(n+1)/2 \geq k$. In turn, (26) follows.

Derivation of Expression (27). Using (1) and (24), $\Pi(k, n-1) - \Pi(k, n) = g\{\beta - (1-q_1)b(k-1, n-1, 1-q_1)/q_2 b(k-1, n-1, q_2)\}$, where g is a positive number. This can be reexpressed as $\Pi(k, n-1) - \Pi(k, n) = g\{\beta - \phi(k+1)\}$. From (A2), in turn, (27) follows.