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Raaj Sah and Jingang Zhao

International Economic Review, Volume 39, Number 3, August 1998, pages 623-634.

ABSTRACT

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Previous working versions include:

"Some envelope theorems for integer and discrete choice variables." Yale University, Department of Economics. Economic Growth Center, Center discussion paper number 598. March 1990.

INTERNATIONAL ECONOMIC REVIEW
Vol. 39, No. 3, August 1998

SOME ENVELOPE THEOREMS FOR INTEGER AND DISCRETE CHOICE VARIABLES*

BY RAAJ SAH AND JINGANG ZHAO†[1]

University of Chicago, U.S.A.
Ohio State University, U.S.A.

The envelope theorem is a genuine workhorse of economic analysis. Typically, this theorem requires that the choice variables be continuous. This paper derives envelope theorems, previously unavailable in the literature, for use with integer and discrete choice variables. Our results, which are intuitive, thus make it possible to use the envelope theorem in a variety of analyses in which the natural description of choice variables is not continuous. Among such choice variables are the number of projects, plants, and a couple's children, as well as binary (yes–no) choices such as labor-force participation, home ownership, and migration.

1. INTRODUCTION

The envelope theorem is a genuine workhorse of economic analysis. It is used in numerous contexts, such as the indirect utility function, the cost function, and the profit function.[2] A limitation of the typically-used envelope theorem is that choice variables in the underlying optimization must necessarily be continuous. This limitation is significant because the natural description of many economic variables is as integers or discrete variables; for example, the number of a company's projects and plants, and a couple's children. This feature is actually more widespread in economics, since it arises in all those situations in which some of the choices are indivisible (e.g., where there are different fixed costs associated with different non-zero levels of choices), or where there are binary (yes versus no) choices, such as labor-force participation, home ownership, and migration.

Because of the seeming difficulties in working with integer choice variables, economists typically represent these variables as continuous. Such representations are often unsatisfactory, and sometimes a source of potential error. For example, a continuous representation is potentially more erroneous for binary choice variables, or for variables with small magnitudes (such as the number of children produced by a couple), than for variables with large magnitudes (such as the number of widgets

* Manuscript received August 1994; final version May 1997.

† E-mail: *r-sah@uchicago.edu; zhao.18@osu.edu*

[1] We thank Alex Ascencios, James Mirrlees, Hal Varian, Martin Weitzman, three anomymous referees, and the editor for their valuable suggestions.

[2] See, for example, Diewert (1982) and Varian (1992). See Dixit and Norman (1980) for initiating an extensive use of the envelope theorem in the analysis of international trade.

produced in a factory). Another important consideration is as follows. The conventional view is that economic analyses become more tractable if a continuous representation is employed for intrinsically discrete variables. However, this is not necessarily true for certain kinds of economic analyses in which a continuous representation may lead to a substantial loss of useful information. In such cases, the analysis may become more intractable or unproductive if a continuous representation is employed.[3]

We present three theorems that ameliorate this limitation. Our main result is Theorem 3, which establishes the envelope property that: the derivative of the extreme value function is equal to the partial derivative of the optimand with respect to the parameter, valued at the optimal point. Theorem 1 provides the necessary and sufficient conditions for the existence of the derivative of the extreme value function. Theorem 2 provides an existence result for one-side derivatives. As will be seen, our envelope theorems can be employed with integer and discrete variables under the usual conditions. Though these results are intuitive, they are not obvious; nor, to our knowledge, have they been reported in the economics literature.

It seems likely, at first glance, that the envelope theorem for integer variables would have appeared in the mathematics literature on topics such as optima, sensitivity analysis, and nonsmooth analysis. However, this is not the case: our theorems are also new to the mathematics literature, except that the sufficiency (but not the necessity) part of Theorem 1 can be derived from a result of Clarke concerning the generalized gradient (Clarke 1983, p. 47). As discussed later, Clarke's arguments require the use of techniques that are likely to be unfamiliar to many economists. Separately, though the term 'envelope theorem' is popular among economists, it appears to be unfamiliar to those mathematicians whose general interests include the conditions for the differentiability or subdifferentiability of the extreme value function.

As discussed earlier, the envelope theorem is a crucial tool of economic analysis. Given the prevalence and importance of integer and discrete choice variables, our results may significantly extend the applicability of this theorem. Our results are based on straightforward arguments, employing only basic calculus.

Note that this paper is concerned only with the envelope theorem for integer and discrete optimization. It does not deal with other issues involving discreteness.[4] The paper is organized as follows. Section 2 presents an informal description of the results. Section 3 describes the main results and their proofs. Section 4 concludes with some extensions.

[3] For example, Sah (1991) established that the number of children produced by a couple decreases if the child mortality rate declines, a pattern that has been observed in numerous empirical studies. Part of the reason why it was difficult to predict this pattern earlier is that the number of children was typically represented by a continuous variable. Thus, realism and tractability go hand in hand in this context.

[4] See Frank (1969) and Scarf (1981, 1986) on production with indivisibilities; and Scarf (1994) and Shapley and Scarf (1974) on equilibrium with indivisible commodities. One might also try to extend the convexity and subgradients of the optimal value function (see Rockafellar 1970 and Clarke 1983) to discrete optimization. See Mount and Reiter (1990, 1996) on modular networks.

2. AN INFORMAL DESCRIPTION OF THE RESULTS

Consider a competitive firm's profit maximization:

$$\max\{\pi(x,p) \equiv px - c(x) | 0 \leq x \leq y\},$$

where x is the quantity of output, p is the fixed price of the output, $y > 0$ is the firm's production capacity, and $c(x)$ is the cost function. Under the standard conditions (for example, $\partial\pi^2/\partial x^2 = -d^2c/dx^2 < 0$, and the optimal choice is an interior solution), both the solution function $x(p)$ and the envelope function $e(p)$ are well defined and are differentiable, where

$$e(p) \equiv \pi(x(p),p) = \max\{\pi(x,p) \equiv px - c(x) | 0 \leq x \leq y\}.$$

The chain rule, $e_p(p) \equiv de/dp = (\partial\pi/\partial x)(dx/dp) + \partial\pi/\partial p$, and the first-order condition (FOC) $\partial\pi/\partial x = 0$, yield the standard envelope theorem

$$e_p(p) = \partial\pi/\partial p|_{x=x(p)} = x(p).$$

That is, the marginal effect of price on the maximum profit is the same as that on the optimand, provided that the latter partial derivative is evaluated at the optimal output.

However, the above theorem cannot be used if the output x is restricted to take only integer values. This is illustrated in the following simple location problem.

EXAMPLE 1. Consider the choice of locating one emergency response team on a busy road that has $k+1$ equi-distant rest areas (see Figure 1). The team can be installed in only one of the rest areas, which are denoted by n, where $n = 0, 1, \ldots, k$. The most likely location for major accidents to occur, denoted by θ, can be anywhere in between $(0 \leq \theta \leq k)$. Suppose that the relevant utility function is: $f(n,\theta) = 7 - (n-\theta)^2$, where $\theta \in [0,k]$ is a given parameter for the decision problem. This utility function is nonlinear and decreasing in the distance between the team location n and the accident location θ, reflecting the nonlinear cost of the response time. Thus, the parametric optimization problem at hand is: $\max\{f(n,\theta) | n \in [0,k]$, and n can take only integer values$\}$. Such formulations also apply to similar problems, like locating fire stations, hospitals, and bank branches.

Let $N(\theta)$ denote the set of values of n that are optimal for a given value of θ. $N(\theta)$ is shown in Figure 1, and it is

$$N(\theta) = \{i\} \text{ if } i - 0.5 < \theta < i + 0.5, \text{ where } i = 0,1,2,\ldots,k$$

$$= \{i-1, i\} \text{ if } \theta = i - 0.5, \text{ where } i = 1,2,\ldots,k.$$

The envelope function is shown in Figure 2 and it is given by

$$e(\theta) = 7 - (i-\theta)^2 \text{ if } i - 0.5 \leq \theta \leq i + 0.5, \text{ where } i = 0,1,\ldots,k.$$

As illustrated in Figure 2, the derivative $e_\theta(\theta)$ does not exist at values of $\theta = i + 0.5$, for $i = 0, 1, \ldots, k-1$.

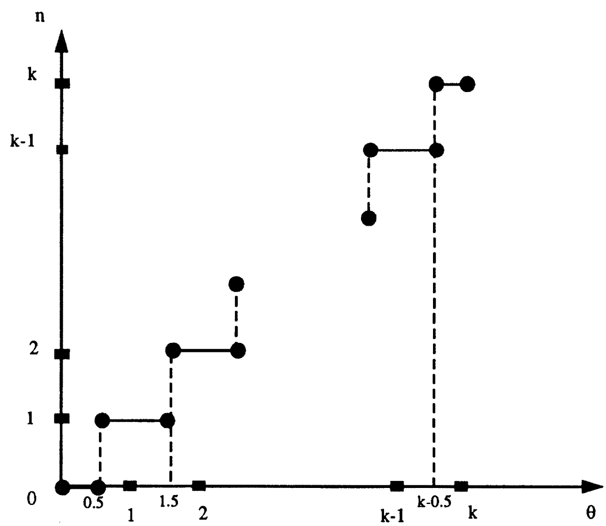

FIGURE 1

THE OPTIMAL SOLUTION CORRESPONDENCE $n(\theta)$.

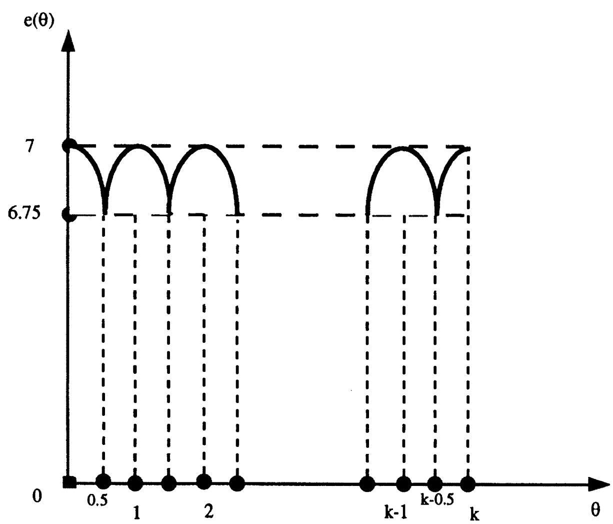

FIGURE 2

THE ENVELOPE FUNCTION $e(\theta)$.

A General Optimization Problem. Now we abstract from specific economic contexts, and consider the general optimization problem (1) below. Unless stated otherwise, *it is assumed throughout the paper that n can take only integer values.*

$$e(\theta) = f(n(\theta), \theta) = \max\{f(n, \theta) | n \in D\}, \tag{1}$$

for each $\theta \in T$; where θ is a continuous scalar parameter; $D = [\underline{\delta}, \bar{\delta}]$ and $T = [\underline{t}, \bar{t}]$ are finite closed intervals; and $n(\theta)$ denotes any value of n that is optimal for a given value of θ. Since D is a finite interval, both $n(\theta)$ and $e(\theta)$ are well-defined.

Given our focus, we are interested in the conditions under which the envelope theorem

$$e_\theta(\theta) = f_\theta(n, \theta)|_{n=n(\theta)} \tag{2}$$

holds, where $f_\theta \equiv \partial f / \partial \theta$. To keep matters simple at present, assume that f is strictly concave in n and differentiable in θ. Then, the discrete optimization problem (1) has two possible outcomes: a unique optimal value of n, and two optimal values of n that are neighboring integers. These two outcomes are illustrated, respectively, at $\theta = 1$ and 1.5 in Figure 1.

When the optimal choice is unique (e.g., $\theta = 1$ in Figures 1 and 2), we show that the derivative $e_\theta(\theta)$ exists and that the standard envelope theorem (2) holds. When there are two optimal choices, denote them by $\bar{n}(\theta)$ and $\underline{n}(\theta)$. We show that $e_\theta(\theta)$ exists if and only if

$$f_\theta(\bar{n}(\theta), \theta) = f_\theta(\underline{n}(\theta), \theta).$$

That is, the derivative of the envelope function exists if and only if the partial derivative of the optimand has the same value at both optimal solutions. In this case, the envelope theorem is

$$e_\theta(\theta) = f_\theta(n, \theta)|_{n=\underline{n}(\theta)} = f_\theta(n, \theta)|_{n=\bar{n}(\theta)}. \tag{3}$$

However, the derivative $e_\theta(\theta)$ does not exist in general (e.g., at $\theta = 0.5$ in Figure 2). From the preceding result, $e_\theta(\theta)$ does not exist if and only if $f_\theta(\bar{n}(\theta), \theta) \neq f_\theta(\underline{n}(\theta), \theta)$. In this case, we show that, under the usual conditions (Assumption 1 in the next section), the right-hand (left-hand) derivative $e_\theta^+(\theta)$ ($e_\theta^-(\theta)$) always exists and that the envelope theorem can be modified as follows: $e_\theta^+(\theta)$ ($e_\theta^-(\theta)$) equals the larger (the smaller) of $f_\theta(\bar{n}(\theta), \theta)$ and $f_\theta(\underline{n}(\theta), \theta)$. Thus, for example, if $f_\theta(\bar{n}(\theta), \theta) > f_\theta(\underline{n}(\theta), \theta)$, then

$$e_\theta^+(\theta) = f_\theta(n, \theta)|_{n=\bar{n}(\theta)}, \quad \text{and} \quad e_\theta^-(\theta) = f_\theta(n, \theta)|_{n=\underline{n}(\theta)}. \tag{4}$$

3. THE MAIN RESULTS

Consider the parametric optimization problem (1), and define

$$N(\theta) \equiv \text{Arg-Max}\{f(n, \theta) | n \in D\} \tag{5}$$

as the set of optimal solutions for each θ. Since D contains a finite number of integers, $N(\theta)$ is nonempty. Thus, $N(\theta)$ can be viewed as a correspondence from the parameter θ to a nonempty set of integers contained in D (i.e., $N: T \to$ nonempty subsets of integers in D). We present our results in a bare-bones mathematical setting, working with Assumption 1 stated below.

ASSUMPTION 1. (i) For each $\theta \in T$, $f(n, \theta)$ is strictly concave in n. Thus,

$$g(n,\theta) > g(n+1,\theta), \text{ where } g(n,\theta) \equiv f(n,\theta) - f(n-1,\theta). \tag{6}$$

(ii) For each $n \in D$, $f(n, \theta)$ is differentiable in the scalar parameter θ. This assumption leads directly to the following lemma:

LEMMA 1. (i) For each $\theta \in T$, $N(\theta)$ contains at least one value, and at most two values. If $N(\theta)$ contains two values, then these two must be neighboring integers. (ii) The correspondence N is upper hemi-continuous[5] in θ. (iii) e is continuous in θ.

PROOF. Part (i) follows from the strict concavity. A solution $n(\theta)$ must satisfy

$$g(n(\theta),\theta) \geq 0 \geq g(n(\theta)+1,\theta). \tag{7}$$

From (6), at least one of the two weak inequalities in (7) must be a strict inequality. Further, (6) and (7) imply that $g(n,\theta) > 0$ for $n < n(\theta)$, and $g(n,\theta) < 0$ for $n > n(\theta) + 1$.

Recall that one of the two weak inequalities in (7) is a strict inequality. Suppose that the first weak inequality in (7) is strict. Then, $N(\theta)$ contains one value, $n(\theta)$, if the second weak inequality in (7) is also strict. On the other hand, $N(\theta)$ contains two values, $n(\theta)$ and $n(\theta)+1$, if the second weak inequality in (7) is an equality. Thus, $N(\theta)$ contains at least one value, and at most two values. In the latter case, the two values are neighboring integers. The same conclusions are obtained if one begins by assuming that the second weak inequality in (7) is strict. This proves part (i) of the lemma.

Parts (ii) and (iii) follow from Berge's maximum theorem. The proof uses the same arguments as those used in proving the upper hemi-continuity of a consumer's optimal choices (see, for example, Section 2 in Debreu 1982). This completes the proof of Lemma 1.[6] Q.E.D.

We now consider the differentiability of $e(\theta)$. Our goal is to find conditions under which the derivative or the one-sided derivatives of $e(\theta)$ exist, and then to derive envelope theorems, given that the choice variable is an integer. Theorem 1 shows that the derivative $e_\theta(\theta)$ exists if and only if the partial derivative $f_\theta(n, \theta)$ has the same value at all optimal values of n. Theorem 2 shows that the one-sided

[5] Upper hemi-continuity is traditionally used for correspondences and upper semi-continuity is reserved for functions. See Novshek (1993) for further discussion.

[6] Note that this lemma requires only the continuity of f in θ but not its differentiability. Thus, part (ii) of Assumption 1 can be weakened to continuity.

derivatives of $e(\theta)$ always exist. Theorem 3 presents three versions of the envelope theorem. Version (a) holds if the derivative $e_\theta(\theta)$ exists, and versions (b) and (c) always hold.

THEOREM 1. *The derivative $e_\theta(\theta)$ exists if and only if $f_\theta(n,\theta)$ has the same value at all optimal values of n. That is, $e_\theta(\theta)$ exists if and only if there exists a constant q, such that $f_\theta(n,\theta)=q$ for all $n\in N(\theta)$.*

PROOF. We first prove the sufficiency part of this theorem. There are two cases: when the optimal n is unique and when it is not.

Case 1. $N(\theta)=\{n^*(\theta)\}$. From Lemma 1, there exists $\delta>0$, such that for all $|\theta-\theta'|<\delta$, $N(\theta')=N(\theta)=\{n^*(\theta)\}$. Thus,

$$e_\theta(\theta)=\lim_{\theta'\to\theta}\frac{e(\theta')-e(\theta)}{\theta'-\theta} \tag{8}$$

$$=\lim_{\theta'\to\theta}\frac{f(n^*(\theta),\theta')-f(n^*(\theta),\theta)}{\theta'-\theta}=f_\theta(n,\theta)|_{n=n^*(\theta)}.$$

The last part of (8) follows from the fact that f is differentiable in θ. Then, $e_\theta(\theta)$ exists.

Case 2. $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$. By assumption we have

$$f_\theta(\underline{n}(\theta),\theta)=f_\theta(\bar{n}(\theta),\theta),\text{ and} \tag{9a}$$

$$e(\theta)=f(\underline{n}(\theta),\theta)=f(\bar{n}(\theta),\theta). \tag{9b}$$

Again, from Lemma 1, there exists $\delta>0$, such that for all $|\theta-\theta'|<\delta$,

$$e(\theta')=\max\{f(\underline{n}(\theta),\theta'),f(\bar{n}(\theta),\theta')\}.$$

Thus, using (9b), $e(\theta')-e(\theta)=\max\{f(\bar{n}(\theta),\theta')-f(\bar{n}(\theta),\theta), f(\underline{n}(\theta),\theta')-f(\underline{n}(\theta),\theta)\}$. In turn,

$$\frac{e(\theta')-e(\theta)}{\theta'-\theta}$$

$$=\max\left\{\frac{f(\bar{n}(\theta),\theta')-f(\bar{n}(\theta),\theta)}{\theta'-\theta},\frac{f(\underline{n}(\theta),\theta')-f(\underline{n}(\theta),\theta)}{\theta'-\theta}\right\}\text{ if }\theta'>\theta;\text{ and}$$

$$=\min\left\{\frac{f(\bar{n}(\theta),\theta')-f(\bar{n}(\theta),\theta)}{\theta'-\theta},\frac{f(\underline{n}(\theta),\theta')-f(\underline{n}(\theta),\theta)}{\theta'-\theta}\right\}\text{ if }\theta'<\theta.$$

Now, evaluate the above expression for $\theta' \to \theta$. The left-hand side is $e_\theta(\theta)$, if it exists. The right-hand side is either $f_\theta(n, \theta)|_{n=\underline{n}(\theta)}$ or $f_\theta(n, \theta)|_{n=\bar{n}(\theta)}$. The preceding two values are the same because of (9a). Thus $e_\theta(\theta)$ exists, and

$$e_\theta(\theta) = f_\theta(n, \theta)|_{n \in N(\theta)}. \tag{10}$$

We now prove the necessity part of the theorem. When $N(\theta) = \{n^*(\theta)\}$, the result is trivially true. When $N(\theta) = \{\underline{n}(\theta), \bar{n}(\theta)\}$, assume, by way of contradiction, that

$$f_\theta(n, \theta)|_{n=\bar{n}(\theta)} > f_\theta(n, \theta)|_{n=\underline{n}(\theta)}. \tag{11}$$

The opposite inequality leads to a similar contradiction. Now, define

$$F(\theta') = \frac{1}{\theta' - \theta}\left[f(\bar{n}(\theta), \theta') - f(\underline{n}(\theta), \theta')\right]. \tag{12}$$

It follows from (11), (12) and (9b) that

$$\begin{aligned} &\lim_{\theta' \to \theta} F(\theta') \\ &= \lim_{\theta' \to \theta} \frac{1}{\theta' - \theta}\{[f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)] - [f(\underline{n}(\theta), \theta') - f(\underline{n}(\theta), \theta)]\} \\ &= f_\theta(n, \theta)|_{n=\bar{n}(\theta)} - f_\theta(n, \theta)|_{n=\underline{n}(\theta)} > 0. \end{aligned} \tag{13}$$

From (12) and (13), there exists $\delta' > 0$, such that $F(\theta') > 0$ if $0 < |\theta - \theta'| < \delta'$. Thus,

$$\begin{aligned} &f(\bar{n}(\theta), \theta') > f(\underline{n}(\theta), \theta') \text{ if } \theta < \theta' \text{ and } 0 < |\theta - \theta'| < \delta', \text{ and} \\ &f(\bar{n}(\theta), \theta') < f(\underline{n}(\theta), \theta') \text{ if } \theta > \theta' \text{ and } 0 < |\theta - \theta'| < \delta'. \end{aligned} \tag{14}$$

From part (b) of Lemma 1, there exists $\delta > 0$, such that $N(\theta') \subseteq \{\underline{n}(\theta), \bar{n}(\theta)\}$ if $|\theta - \theta'| < \delta$. From (14), therefore, for $0 < |\theta - \theta'| < \min(\delta, \delta')$, the following holds:

$$N(\theta') = \{\bar{n}(\theta)\} \text{ if } \theta' > \theta, \text{ and } N(\theta') = \{\underline{n}(\theta)\} \text{ if } \theta' < \theta.$$

Thus,

$$\begin{aligned} e_\theta^+ &= \lim_{\theta' \to \theta;\, \theta' > \theta} \frac{e(\theta') - e(\theta)}{\theta' - \theta} \\ &= \lim_{\theta' \to \theta;\, \theta' > \theta} \frac{f(\bar{n}(\theta), \theta') - f(\bar{n}(\theta), \theta)}{\theta' - \theta} = f_\theta(n, \theta)|_{n=\bar{n}(\theta)}. \end{aligned}$$

Similarly $e_\theta^-(\theta) = f_\theta(n, \theta)|_{n=\underline{n}(\theta)}$. Thus, from (11), $e_\theta^+(\theta) > e_\theta^-(\theta)$, which contradicts the assumption that $e_\theta(\theta)$ exists. Q.E.D.

Note that the sufficiency, but not the necessity, part of Theorem 1 can be derived from Clarke's result concerning the generalized gradient (Clarke 1983, p. 47). However, our proof does not need to employ the set of sophisticated techniques that Clarke has developed for more general purposes.

Next we show that both one-sided derivatives always exist.

THEOREM 2. *The one-sided derivatives of the extreme value function always exist. That is, both* $e_\theta^+(\theta)$ *and* $e_\theta^-(\theta)$ *always exist.*

PROOF. If $N(\theta)=\{n^*(\theta)\}$, or if $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)=f_\theta(\underline{n}(\theta),\theta)$, then from Theorem 1, $e_\theta(\theta)$ exists, and thus both $e_\theta^+(\theta)$ and $e_\theta^-(\theta)$ exist. Now consider the case in which $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)\neq f_\theta(\underline{n}(\theta),\theta)$. It follows from the necessity part of the proof of Theorem 1 that

$$(15)\qquad e_\theta^+(\theta)=f_\theta(n,\theta)|_{n=\bar{n}(\theta)} \text{ and } e_\theta^-(\theta)=f_\theta(n,\theta)|_{n=\underline{n}(\theta)} \text{ if } f_\theta(\bar{n}(\theta),\theta)>f_\theta(\underline{n}(\theta),\theta).$$

Analogously, it can be shown that

$$(16)\qquad e_\theta^+(\theta)=f_\theta(n,\theta)|_{n=\underline{n}(\theta)} \text{ and } e_\theta^-(\theta)=f_\theta(n,\theta)|_{n=\bar{n}(\theta)} \text{ if } f_\theta(\bar{n}(\theta),\theta)<f_\theta(\underline{n}(\theta),\theta).$$

Thus, both $e_\theta^+(\theta)$ and $e_\theta^-(\theta)$ exist. Q.E.D.

Now suppose $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)\neq f_\theta(\underline{n}(\theta),\theta)$. Let $N^+(\theta)(N^-(\theta))$ denote the member of $N(\theta)$ at which $f_\theta(n,\theta)$ is larger (smaller). That is,

$$N^+(\theta)=\{\bar{n}(\theta)\} \text{ if } f_\theta(\bar{n}(\theta),\theta)>f_\theta(\underline{n}(\theta),\theta),$$

$$N^+(\theta)=\{\underline{n}(\theta)\} \text{ if } f_\theta(\bar{n}(\theta),\theta)<f_\theta(\underline{n}(\theta),\theta), \text{ and}$$

$$N^-(\theta)=N(\theta)/N^+(\theta).$$

Using these notations, we can summarize our previous results into the following four Cases:

(i) If $N(\theta)=\{n^*(\theta)\}$, then $e_\theta(\theta)=f_\theta(n,\theta)|_{n=n^*(\theta)}$.

(ii) If $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)=f_\theta(\underline{n}(\theta),\theta)$, then

$$e_\theta(\theta)=f_\theta(n,\theta)|_{n=\bar{n}(\theta)}=f_\theta(n,\theta)|_{n=\underline{n}(\theta)}.$$

(iii) If $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)>f_\theta(\underline{n}(\theta),\theta\}$, then

$$e_\theta^+(\theta)=f_\theta(n,\theta)|_{n=\bar{n}(\theta)}=f_\theta(n,\theta)|_{n\in N^+(\theta)}, \text{ where } N^+(\theta)=\{\bar{n}(\theta)\}.$$

$$e_\theta^-(\theta)=f_\theta(n,\theta)|_{n=\underline{n}(\theta)}=f_\theta(n,\theta)|_{n\in N^-(\theta)}, \text{ where } N^-(\theta)=\{\underline{n}(\theta)\}.$$

(iv) If $N(\theta)=\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $f_\theta(\bar{n}(\theta),\theta)<f_\theta(\underline{n}(\theta),\theta)$, then

$$e_\theta^+(\theta)=f_\theta(n,\theta)|_{n=\underline{n}(\theta)}=f_\theta(n,\theta)|_{n\in N^+(\theta)},\text{ where } N^+(\theta)=\{\underline{n}(\theta)\}.$$

$$e_\theta^-(\theta)=f_\theta(n,\theta)|_{n=\bar{n}(\theta)}=f_\theta(n,\theta)|_{n\in N^-(\theta)},\text{ where } N^-(\theta)=\{\bar{n}(\theta)\}.$$

THEOREM 3. (a) *If* $e_\theta(\theta)$ *exists, then* $e_\theta(\theta)=f_\theta(n,\theta)|_{n\in N(\theta)}$. (b) $e_\theta^+(\theta)=f_\theta(n,\theta)|_{n\in N^+(\theta)}$. (c) $e_\theta^-(\theta)=f_\theta(n,\theta)|_{n\in N^-(\theta)}$.

PROOF. Part (a) was established in the course of deriving (8) and (10). Parts (b) and (c) were established respectively in Cases (iii) and (iv) preceding this theorem. Q.E.D.

An Alternative Proof of Parts (b) and (c) of Theorem 3. For use in an extension presented in the next section, we now provide an alternative and somewhat more general proof of parts (b) and (c) of Theorem 3. Redefine sets $N^+(\theta)$ and $N^-(\theta)$ as follows: $n\in N^+(\theta)$ (respectively, $n\in N^-(\theta)$) if and only if there are sequences $\{\theta_t\}$ and $\{n_t\}$, such that for all t, $\theta_t>\theta$ (respectively, $\theta_t<\theta$), $n_t\in N(\theta_t)$, and $\theta_t\to\theta$, $n_t\to n$, as $t\to\infty$. The nonemptiness of $N(\theta)$ leads to the nonemptiness of $N^+(\theta)$ and $N^-(\theta)$. From part (ii) of Lemma 1, both $N^+(\theta)$ and $N^-(\theta)$ are subsets of $N(\theta)$. From Assumption 1, $N^+(\theta)$ and $N^-(\theta)$, just defined, are the same as those defined earlier in the paper. That is, $N^+(\theta)$ ($N^-(\theta)$) denotes the member of $N(\theta)$ at which $f_\theta(n,\theta)$ is larger (smaller). We prove here only part (b) of Theorem 3 because the proof of part (c) is analogous.

Let $n\in N^+(\theta)$. Since $N^+(\theta)\subseteq N(\theta)$, $n=\bar{n}(\theta)$ or $\underline{n}(\theta)$. Without loss of generality, assume that $n=\bar{n}(\theta)$. Therefore, there exist $\{\theta_t\}$ and $\{n_t\}$, such that for all t, $\theta_t>\theta$, $n_t\in N(\theta_t)$, and $\theta_t\to\theta$, $n_t\to\bar{n}(\theta)$, as $t\to\infty$. From part (ii) of Lemma 1, $N(\theta_t)\subseteq N(\theta)$, for t sufficiently large. Since $n_t\in N(\theta_t)\subseteq\{\underline{n}(\theta),\bar{n}(\theta)\}$ and $n_t\to\bar{n}(\theta)$, as $t\to\infty$, $\bar{n}(\theta)\in N(\theta_t)$ must hold for all t that are sufficiently large. Hence, there exists a subsequence $\{\theta_s\}$ of $\{\theta_t\}$, such that $n_s=\bar{n}(\theta)$ for all s. In turn, from the existence of $e_\theta^+(\theta)$ and f_θ, we have

$$\begin{aligned} e_\theta^+(\theta) &= \lim_{s\to\infty}\frac{e(\theta_s)-e(\theta)}{\theta_s-\theta} \\ &= \lim_{s\to\infty}\frac{f(\bar{n}(\theta),\theta_s)-f(\bar{n}(\theta),\theta)}{\theta_s-\theta}=f_\theta(n,\theta)|_{n=\bar{n}(\theta)}. \end{aligned}$$

This proves part (b). Q.E.D.

Note that, for the purposes of this paper, Assumption 1 is a sufficient but not a necessary condition. One can construct examples that exhibit the envelope properties but fail to satisfy the two conditions stated in this assumption.[7]

4. SOME EXTENSIONS

Keeping mathematical details to the necessary minimum, we have derived a set of envelope theorems. The analysis entailed a single choice variable and a single

[7] For example, $f(n,\theta)=-(n-\theta)^2$ if $1\le n\le 2$, and $f(n,\theta)=-18$ otherwise; where $\theta\in T=[1,2]$, and $n\in D=[-10,10]$.

parameter. We showed that, under the usual conditions, our envelope theorems (including the one-sided versions) can always be employed with integer variables.

In the rest of the paper we describe three extensions without providing proofs. Note that the envelope theorem typically deals with the effects of an infinitely small change in θ. In the first extension, we derive envelope-like properties even if the change in θ is not small. For additional discussion of problems of this type, see Anderson and Takayama (1979).

Extension 1. Suppose the parameter is changed from θ to θ'. Then, from (1),

$$e(\theta') - e(\theta) = f(n(\theta'), \theta') - f(n(\theta), \theta) \geq f(n(\theta), \theta') - f(n(\theta), \theta). \tag{17}$$

If the change in θ leads to a nontrivial change in $n(\theta)$, then

$$e(\theta') - e(\theta) > f(n(\theta), \theta') - f(n(\theta), \theta). \tag{18}$$

By a nontrivial change we mean that a choice of n, say $n(\theta)$, that was optimal at θ is not optimal at θ' (i.e., $n(\theta) \notin N(\theta')$, where $N(\theta')$ is the optimal set at θ'). Thus, *the effect of a change in parameter θ (large or small) on the envelope function is not smaller than its effect on the optimand when the choice variable is kept unchanged.* Moreover, the former is strictly larger than the latter if the change in θ leads to a nontrivial change in the optimal choice $n(\theta)$.

Extension 2. Our results can be generalized to many choice variables and many parameters, instead of one each, as examined earlier. We shall discuss two cases. First, suppose n is a scalar, and $\theta = (\theta_1, \ldots, \theta_I)$ is a vector of parameters. Then all of the previous results hold for each of the parameters. Next, suppose θ is a scalar but $n = (n_1, \ldots, n_J)$ is a vector (the generalization in which θ as well as n are vectors is straightforward). Each element of n can take only restricted discrete values. Assume that f is strictly concave in vector n, and n is restricted to take integer values in a closed bounded convex set in $\mathbb{R}^J$. Define $n(\theta)$ as an optimal vector. Define the optimal sets $N(\theta)$, $N^+(\theta)$ and $N^-(\theta)$ accordingly (as in the alternative proof of parts of Theorem 3). Then, parts (ii) and (iii) of Lemma 1 continue to hold as stated earlier. Part (i) of Lemma 1 becomes the following: $N(\theta)$ contains at most 2^J vectors. Theorems 1, 2 and 3 hold with corresponding modifications.

Extension 3. Our theorems can be extended to mixed programming problems such as:[8]

$$e(\theta) = f(x(\theta), n(\theta); \theta) = \max\{f(x, n; \theta) | n \in D \text{ and } x \in X\},$$

where the newly-added choice variable x can take continuous values; $X = [\underline{x}, \bar{x}]$, $D = [\underline{\delta}, \bar{\delta}]$ and $T = [\underline{t}, \bar{t}]$ are three finite closed intervals; and $(x(\theta), n(\theta))$ denotes a choice that is optimal corresponding to each θ. Assume that: (i) f is strictly concave in (x, n) and differentiable in θ, and (ii) an optimal solution $(x(\theta), n(\theta))$ is an

[8] This extension was motivated by a referee's comments.

interior point of the feasible region (note that the standard envelope theorem is based on a unique and interior solution). Then the optimal set is either unique or it contains two points, that is, $N(\theta) = \{x^*(\theta), n^*(\theta)\}$, or $\{(x^*(\theta), \bar{n}(\theta)); (x^*(\theta), \underline{n}(\theta))\}$, where the optimal value of the continuous choice variable, $x^*(\theta)$, is unique. All three of our theorems hold with slight modifications.

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