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# Some results for the comparative statics of steady states of higher-order discrete dynamic systems☆

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**Abstract**

Higher-order discrete dynamic systems arise naturally in many economic models in which the problem at hand requires an explicit treatment of dynamics involving lags of more than one period. In studying such models, one type of analysis that economists are often interested in is the assessment of the signs and the magnitudes of the effects of a change in the parameters of the dynamic system on the stable steady states of the system. This paper presents some results for such comparative statics analysis. Brief remarks explain the results and illustrate their potential economic usefulness. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Consider autonomous discrete dynamic systems represented by the following set of $K$ equations:

$$y(T) \equiv \begin{bmatrix} y_1(T) \\ \vdots \\ y_j(T) \\ \vdots \\ y_K(T) \end{bmatrix} = \begin{bmatrix} f_1(y(T-1),\ \ldots, y(T-L),\ \theta_1) \\ \vdots \\ f_j(y(T-1),\ \ldots, y(T-L),\ \theta_j) \\ \vdots \\ f_K(y(T-1),\ \ldots,\ y(T-L),\ \theta_K) \end{bmatrix}. \tag{1}$$

Here, $L$ denotes the order of the system. That is, at least one equation has an $L$-period lag but no equation has more than an $L$-period lag. $K \geq 1$ and $L \geq 1$. $T$ denotes time. $\theta_j$ denotes a parameter that affects the $j$th equation. The generalization to the case in which a parameter affects more than one structural equation in (1) is straightforward.

Systems such as (1) arise naturally in economic models in which the problem at hand requires an explicit treatment of dynamics involving lags of more than one period. In models of firm behavior, a firm's current optimal choices often depend on its choices in earlier periods because of technological or economic relationships that link past and future choices. Examples include 'time to build' models (see Kydland and Prescott, 1982) and models of firm behavior in which there are adjustment costs associated with changes in the workforce and in other choice variables (see Sargent, 1987). Another example is a set of models that study the consequences of gestation and maturation lags on optimal herd production and inventory decisions (Rosen, 1987; Rosen et al., 1994). In such models, a firm's current optimal choices, described explicitly or in reduced form, yield dynamic systems such as (1). Yet another example is an analysis of patterns of crime in which informational relationships are a source of multi-period dynamics (Sah, 1991a); for example, an individual learns from his own past experiences as well as of those with whom he interacts. In this analysis, the crime participation rates in different societal groups are derived by aggregating individual decisions in each period (e.g., whether or not to be a criminal in that period), and these rates are affected by the past crime participation rates in different groups. Depending on the context, a dynamic system may have stochastic elements, but an analysis of its deterministic counterpart is often crucial in this case as well (Sargent, 1987, Chapter XI).

When dealing with dynamic systems such as (1), one type of analysis that economists are often interested in is the assessment of the signs and the magnitudes of the effects of a change in the system's parameters on its stable steady states. A reason for the interest in such comparative statics analysis is that it can help highlight some of the qualitative aspects of the behavior of a dynamic system. Let the $(K \times 1)$ vector $Y$ denote a stable steady state value of $y$ (the definition of a stable steady state is provided later). At this steady state, system (1) can be written in reduced form as

$$Y_j = F_j(Y, \theta_j) \quad \text{for } j = 1 \text{ to } K. \tag{2}$$

Let $\mathrm{d}Y_j/\mathrm{d}\theta_k$ denote the derivative of $Y_j$ with respect to a sustained small change in parameter $\theta_k$.[1] This paper presents some results on the signs and the magnitudes of the $\mathrm{d}Y_j/\mathrm{d}\theta_k$. To my knowledge, these results have not been previously reported in the literature.

A potential source of knowledge for comparative statics analysis is the fact that, since the steady states under consideration are stable, they satisfy some stability criteria. However, unlike an important body of the literature that, for a variety of purposes, deals with stability criteria,[2] we do not focus on stability criteria. Our interest in stability criteria is limited here to using some of them to derive the comparative statics results presented in this paper.

Section 2 describes the preliminaries, including the definition of stability. Section 3 presents the results, as well as some brief remarks that explain the results and illustrate their potential economic usefulness.

## 2. Preliminaries

Let $g^1, \ldots, g^L$ denote the $(K \times K)$ gradient matrices of (1). That is, if the $(K \times 1)$ vector-valued function $f$ describes the right-hand side of (1), then $g^\ell \equiv \partial f/\partial y(T-\ell)$. Define $h_k \equiv \partial f_k(y(T-1), \ldots, y(T-L), \theta_k)/\partial\theta_k$. Assume that the $h_k \neq 0$, because our interest is in parameter changes that have nontrivial effects. In the rest of the paper, the $g^\ell$ and the $h_k$ are evaluated at the steady state, $Y$, under consideration. Define the $(K \times K)$ matrix $G \equiv \partial F/\partial Y$, where the $(K \times 1)$ vector-valued function $F$ describes the right-hand side of (2). Then, it follows from (1) and (2) that $G = \Sigma_\ell g^\ell$. Define $M \equiv I_K - G$, where $I_K$ is the identity matrix of order $K$. Let $\mathrm{Det}(M)$ denote the determinant of $M$. Let $C_{kj}$ denote the co-factor corresponding to the $(k \times j)$ element of $M$. Then, a differentiation of (2) with respect to $\theta_k$ yields the comparative statics expressions:

$$\mathrm{d}Y_j/\mathrm{d}\theta_k = C_{kj}h_k/\mathrm{Det}(M). \tag{3}$$

[1] It is assumed that the $\mathrm{d}Y_j/\mathrm{d}\theta_k$ and the other derivatives to be used later are defined in the vicinity of the steady state under consideration. The convention concerning indices used in this paper is that, unless stated otherwise, $i = 1$ to $K$, $j = 1$ to $K$, $k = 1$ to $K$, and $\ell = 1$ to $L$. Further, a nonnegative matrix means that each element of the matrix is nonnegative.

[2] For example, see Barnett and Storey (1970, pp. 115–117), Conlisk (1973), Gandolfo (1980, pp. 108–115, 136–139), Harriff et al. (1980), Murata (1977, pp. 96–101), Quirk and Ruppert (1965), and references therein.

Some interpretations of the above expressions, which are especially useful in economic analysis, are as follows. Recall that $h_k$ is the derivative of $f_k$ with respect to $\theta_k$, evaluated at the pre-change value of the variables. Hence, $h_k$ can be interpreted as the *first-round effect* of a small change in $\theta_k$. On the other hand, $\mathrm{d}Y_j/\mathrm{d}\theta_k$ is the *final steady-state effect* of a sustained small change in $\theta_k$ on different variables. Further, in system (1), the first-round effect of a change in parameter $\theta_k$ is felt only on the $k$th variable; other variables are affected indirectly by the dynamic effects. Hence, $\mathrm{d}Y_k/\mathrm{d}\theta_k$ can be interpreted as the *direct steady-state effect* of a change in parameter $\theta_k$. Analogously, for $j \neq k$, the $\mathrm{d}Y_j/\mathrm{d}\theta_k$ can be interpreted as the *indirect steady-state effects* on different variables.

The steady states that this paper considers are those that are the 'sinks' of the dynamic system. Let $A$ denote a 'companion matrix' to system (1).[3] Let $r(A)$ denote the spectral radius of $A$; that is, the maximum absolute value of the eigenvalues of $A$. Then, a sink is a steady state at which $r(A) < 1$, and this steady state is asymptotically stable (Hirsch and Smale, 1974, p. 280). If the dynamic system is slightly perturbed from a sink, it converges to the same steady state (Hirsch and Smale, 1974, p. 181).

The following lemma (Sah, 1991b) plays a central role in the analysis below.

*Lemma. At a stable steady state of* (1),

$$\mathrm{Det}(M) > 0. \tag{4}$$

## 3. Results for comparative statics

The first comparative statics result presented in this section is for the general dynamic system described in (1). The other results hold when the steady state under consideration satisfies specific properties described later.

*Result 1*

$$\mathrm{sgn}\{\mathrm{d}Y_j/\mathrm{d}\theta_k\} = \mathrm{sgn}\{C_{kj}h_k\}. \tag{5}$$

[3] This matrix is obtained as follows. Using standard procedures (e.g., Chow, 1975, p. 22), one can transform (1) into a first-order system. Let $z(T) = w(z(T-1))$ be the first-order system, where $z(T)$ is a $(KL \times 1)$ vector and $w$ is a $(KL \times 1)$ vector-valued function. Then, $A \equiv \partial w(z(T-1))/\partial z(T-1)$. Let $O_K$ denote a $(K \times K)$ null matrix. Thus,

$$A = \begin{bmatrix} g^1 & g^2 & \cdots & g^{L-1} & g^L \\ I_K & O_K & \cdots & O_K & O_K \\ \vdots & & & & \\ O_K & O_K & \cdots & I_K & O_K \end{bmatrix}.$$

This result follows directly from (3) and (4). It markedly simplifies the assessment of the signs of comparative statics expressions. This is because, given (4), we do not need to have other sources of knowledge concerning the sign of the determinant of $M$.

As an illustration of the use of Result 1, consider a simple two-equation second-order system: $y_1(T) = f_1(y_2(T-1), \theta_1)$ and $y_2(T) = f_2(y_1(T-2), \theta_2)$, for which one or more stable steady states exist. Let $g^{\ell}_{ij}$ denote the $(i \times j)$ element of $g^{\ell}$. Here, the only non-zero elements of the gradient matrices are $g^1_{12}$ and $g^2_{21}$. Also, $\mathrm{Det}(M) = 1 - g^1_{12} g^2_{21}$. Suppose that we know the signs of $g^1_{12}$ and $g^2_{21}$, and that both of them have the same sign (either both positive or both negative), but that we do not know their magnitudes. Suppose that we also know the signs of $h_1$ and $h_2$. Clearly, the information that we have is substantial. Yet, this information is by itself insufficient to predict the sign of even one of the $\mathrm{d}Y_j/\mathrm{d}\theta_k$. In contrast, the result in (5) permits us to predict the signs of all of the $\mathrm{d}Y_j/\mathrm{d}\theta_k$. It can be verified that $\mathrm{sgn}\{\mathrm{d}Y_1/\mathrm{d}\theta_1\} = \mathrm{sgn}\{h_1\}$, $\mathrm{sgn}\{\mathrm{d}Y_2/\mathrm{d}\theta_1\} = \mathrm{sgn}\{g^2_{21} h_1\}$, $\mathrm{sgn}\{\mathrm{d}Y_1/\mathrm{d}\theta_2\} = \mathrm{sgn}\{g^1_{12} h_2\}$, and $\mathrm{sgn}\{\mathrm{d}Y_2/\mathrm{d}\theta_2\} = \mathrm{sgn}\{h_2\}$.

Next, consider the steady states that satisfy the property that:

$$\text{The matrices } g^{\ell} \text{ are nonnegative.} \tag{6}$$

This property arises in dynamic systems with nonnegative feedbacks. Such systems are important in several areas of research. For example, Arthur (1988, 1990) and David (1988) provide an overview of many economic models, including those dealing with the dynamics of technological innovation and change (for instance, current innovations make future innovations more likely), in which nonnegative feedbacks are central. In the analysis of crime mentioned earlier (Sah, 1991a), the past crime participation rates of different societal groups exert nonnegative feedbacks on the current crime participation rates.

*Result 2.* If the steady state satisfies property (6), then

$$\mathrm{sgn}\{\mathrm{d}Y_k/\mathrm{d}\theta_k\} = \mathrm{sgn}\{h_k\};$$

$$\text{and } \mathrm{sgn}\{\mathrm{d}Y_j/\mathrm{d}\theta_k\} = \text{zero or } \mathrm{sgn}\{h_k\} \text{ for } j \neq k. \tag{7}$$

$$|\mathrm{d}Y_k/\mathrm{d}\theta_k| \geq |h_k|. \tag{8}$$

This result as well as those presented later are proven in the Appendix. There are several possible variations of Result 2. For instance, if (6) is satisfied, and if all the elements of $g^{\ell}$ are positive for at least one $\ell$, then the following stronger version of (7) and (8) is obtained:

$$\mathrm{sgn}\{\mathrm{d}Y_j/\mathrm{d}\theta_k\} = \mathrm{sgn}\{h_k\} \quad \text{and} \quad |\mathrm{d}Y_k/\mathrm{d}\theta_k| > |h_k|. \tag{9}$$

The interpretation of (9) is straightforward: (i) the final effect of a parameter change on any variable has the same sign as that of the first-round effect of the parameter change, and (ii) the direct steady-state effect of a parameter change on a variable has a magnitude larger than that of the first-round effect of the parameter change. The interpretation of (7) and (8) is analogous.

Luenberger (1979, p. 212, Theorem 1) presents a result for first-order linear systems that is a special case of (7). This can be seen as follows. A linear version of system (1) is: $y(T) = \Sigma_\ell g^\ell y(T-\ell) + \theta$, where the $g^\ell$ are matrices of constant coefficients, and $\theta$ is a $(K \times 1)$ vector whose $k$th element is $\theta_k$. From (7), $\mathrm{d}Y_j/\mathrm{d}\theta_k \geq 0$, if property (6) is satisfied. The result just stated is obtained by Luenberger for first-order systems (that is, for $\ell = 1$), under the additional restriction that the elements of $\theta$ are nonnegative.

We now consider the steady states that satisfy the property that:

The rows of matrix $G$ are identical to one another. (10)

One way to understand the economic relevance of this property is as follows. Suppose that the functions $f_j$ are the same for all of the equations in (1), and that we are interested in examining the effects of a change in a parameter, in the vicinity of the case in which all of the functions have the same set of parameter values. Then, the expressions in (3) will be evaluated using property (10).[4] This property arises, for example, in a part of the analysis of crime noted earlier. There, a question is: what is the nature of the effects or 'spill-overs' of a change in the parameters faced by one societal group on various groups' crime participation rates, in the vicinity of the case in which all of the groups are identical? Analogous issues are potentially relevant in the context of spill-overs of one country's (or region's) policies upon others, under the simplification that the set of countries under consideration are otherwise homogenous. For brevity in the analysis below, let the scalar $G_k$ denote the common element of the $k$th column of $G$, when (10) is satisfied.

*Result 3.* If the steady state satisfies property (10), then

$$\mathrm{d}Y_k/\mathrm{d}\theta_k = \{G_k/(1 - \Sigma_i G_i) + 1\}h_k, \quad \text{and} \tag{11}$$

$$\mathrm{d}Y_j/\mathrm{d}\theta_k = \{G_k/(1 - \Sigma_i G_i)\}h_k \quad \text{for } j \neq k. \tag{12}$$

It follows that

$$\frac{\mathrm{d}Y_k}{\mathrm{d}\theta_k} - \frac{\mathrm{d}Y_j}{\mathrm{d}\theta_k} = h_k \quad \text{for } j \neq k. \tag{13}$$

[4] It is readily seen that property (10) is weaker than the assumptions just described. If $g^\ell_{i\cdot}$ denotes the $i$th row of $g^\ell$, then these assumptions imply that $g^\ell_{i\cdot} = g^\ell_{j\cdot}$ for all $i$ and $j$. Hence, since $G = \Sigma_\ell g^\ell$, property (10) is satisfied, that is, $G_{i\cdot} = G_{j\cdot}$ for all $i$ and $j$. However, property (10) can be satisfied even if the $g^\ell_{i\cdot}$ are different for all $i$.

Eqs. (11) and (12) provide closed-form expressions for comparative statics. The interpretation of (13) is that the difference between the change in the steady-state value of a directly affected variable and that of *any* indirectly affected variable equals the first-round effect of a parameter change. Obviously, expressions (13), and (14) to be presented below, are meaningful only if $K \geq 2$.

*Result 4.* If the steady-state satisfies property (10) and $G_k \geq 0$, then (i) expressions (7) and (8) hold, and (ii)

$$|\mathrm{d}Y_k/\mathrm{d}\theta_k| - |\mathrm{d}Y_j/\mathrm{d}\theta_k| = |h_k| > 0 \quad \text{for } j \neq k. \tag{14}$$

Recall conclusions (7) and (8). Note that the sufficient conditions for these conclusions to hold are different in Results 2 and 4. Unlike in Result 4, property (10) is not a condition in Result 2. On the other hand, condition (6) in Result 2 is stronger than the condition $G_k \geq 0$ in Result 4. This is because the former condition implies the latter but not vice versa. Further, analogous to the first part of Result 4, it can be shown that expression (9) holds if (10) is satisfied and $G_k > 0$.

The interpretation of (14) is intuitive. The difference between the magnitude of the change in the steady-state value of a directly affected variable and that of any indirectly affected variable equals the magnitude of the first-round effect of a parameter change. Accordingly, the change in the steady-state value of a directly affected variable has a larger magnitude than that of any indirectly affected variable.

## Appendix

*Proof of Result 2.* This proof relies on the following theorem of Bear (1963, p. 526); see Murata (1977, p. 99) for a convenient statement:

$$\text{If } A \text{ is nonnegative and } r(A) < 1, \text{ then } r(G) < 1. \tag{A.1}$$

Property (6) and the structure of matrix $A$ in footnote 3 imply that $A$ is a nonnegative matrix. Hence, from (A.1) and the definition of a sink, $r(G) < 1$. The preceding inequality implies that $I_K + G + G^2 + \cdots$ converges to $M^{-1}$ (Murata, 1977, p. 85). Let $C$ denote the $(K \times K)$ matrix whose $(k \times j)$ element is $C_{kj}$. Let $C^{\mathrm{t}}$ denote the transpose of $C$. Since $C^{\mathrm{t}} = \mathrm{Det}(M)M^{-1}$, it follows that $C^{\mathrm{t}} = \mathrm{Det}(M)[I_K + G + G^2 + \cdots]$. Since $G$ is a nonnegative matrix, (4) and the preceding relationship yield $C_{kk} \geq \mathrm{Det}(M) > 0$ and $C_{kj} \geq 0$. In turn, (7) and (8) follow from (3).

*Proof of Results 3 and 4.* These results follow from a series of matrix operations made possible by the simple structure of $G$ implied by (10). For brevity, define

$\phi \equiv \Sigma_i G_i$. We first show that (10) yields the following identities:

$$\mathrm{Det}(M) = 1 - \phi, \tag{A.2}$$

$$C_{kk} = 1 - \phi + G_k, \tag{A.3}$$

$$C_{kj} = G_k \quad \text{for } j \neq k. \tag{A.4}$$

To prove (A.2), define (i) vector $\alpha$ as the $(K \times 1)$ vector with unity elements, (ii) vector $e_k$ as the $k$th column of the identity matrix $I_K$, (iii) vector $M_k$ as the $k$th column of matrix $M$, and (iv) $\mu \equiv \Sigma_k M_k$. In matrix $M$, add to column $M_1$, each of the columns from $M_2$ to $M_K$. The resulting matrix is $[\mu, M_2, \ldots, M_K]$. Noting that each element of vector $\mu$ is $1 - \phi$, we obtain

$$\mathrm{Det}(M) = (1 - \phi)\mathrm{Det}(T), \tag{A.5}$$

where matrix $T \equiv [\alpha, M_2, \ldots, M_K]$. Next, in matrix $T$, multiply the first column by $G_k$ and add it to the $k$th column. Repetition of this step for $k = 2$ to $K$ yields the matrix $[\alpha, e_2, \ldots, e_K]$. The determinant of the last matrix is unity. Thus, (A.2) follows from (A.5).

Next, let $B_{kj}$ denote the matrix obtained by deleting the $k$th row and the $j$th column of $M$. That is,

$$C_{kj} = (-1)^{k+j}\mathrm{Det}(B_{kj}). \tag{A.6}$$

Then, the same method that was used to prove (A.2) yields $\mathrm{Det}(B_{kk}) = 1 - \Sigma_{i \neq k} G_i$. In turn, (A.3) follows from (A.6).

To prove (A.4), first consider the case where $j < k$. In matrix $B_{kj}$, subtract the $j$th row from each of the other rows. Call the resulting matrix $S$. Expand the determinant of $S$ along its $(k-1)$st column. This yields $\mathrm{Det}(B_{kj}) = \mathrm{Det}(S) = (-1)^{k+j} G_k$. Thus, (A.4) follows from (A.6). An analogous proof yields (A.4) for the case where $j > k$. Results 3 and 4 follow from (3), (4) and (A.2)–(A.4).

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