Wealth-loss to investors from taxation of investment returns

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ABSTRACT

Since most countries impose significant levels of taxes on the returns from investments, it is important to understand the nature of the burden that such taxation imposes upon investors. This paper derives and analyzes expressions for the tax-burden imposed on an investor in terms of the loss of his initial wealth. These previously-unavailable explicit expressions for the loss are intuitive and informationally highly parsimonious. Surprisingly, the loss does not depend on the nature of the investor's risk aversion, or on any aspect of the risky assets (e.g., the number of these assets, or their distribution of returns). I also show that: (i) any given tax on investment returns is identical in its burden on investors to a proportional tax on their initial wealths, (ii) the latter tax rate depends solely on the former tax rate and the riskless return, and hence (iii) the latter tax rate is the same for all investors regardless of their characteristics and the set of risky assets available to them.

WEALTH-LOSS TO INVESTORS FROM TAXATION OF INVESTMENT RETURNS

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Since most countries impose significant levels of taxes on the returns from investments, it is important to understand the nature of the burden that such taxation imposes upon investors. This paper derives and analyzes expressions for the tax-burden imposed on an investor in terms of the loss of his initial wealth. These previously-unavailable explicit expressions for the loss are intuitive and informationally highly parsimonious. Surprisingly, the loss does not depend on the nature of the investor's risk aversion, or on any aspect of the risky assets (e.g., the number of these assets, or their distribution of returns). I also show that: (i) any given tax on investment returns is identical in its burden on investors to a proportional tax on their initial wealths, (ii) the latter tax rate depends solely on the former tax rate and the riskless return, and hence (iii) the latter tax rate is the same for all investors regardless of their characteristics and the set of risky assets available to them.

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I. INTRODUCTION

Consider an investor whose investment opportunity set consists of one riskless and many risky assets. Suppose that a proportional tax (or subsidy) is imposed on the return from his investments. This paper presents expressions for the resulting tax-burden on the investor in terms of the loss (which can be positive or negative) of his initial wealth. These expressions use the concepts of "equivalent variation" and "compensating variation." These familiar concepts have been employed extensively in deterministic contexts in which an individual faces a change in one or more parameters that affect his choices. Here, these concepts are adapted for use in situations in which an investor allocates his initial wealth across different assets, with stochastic outcomes, and faces a tax on the investment returns.

The expressions for the loss to the investor are intuitive and informationally highly parsimonious. Surprisingly, they do not require knowledge of the nature of the investor's risk-aversion, or of any aspect of the risky assets (for example, how many risky assets are available to the investor, or what is the distribution of the returns from these assets). Put differently, two investors with the same initial wealth, for whom the riskless return is the same, will incur the same tax-burden from a given tax, even if they exhibit different kinds of risk-aversion, and the sets of risky assets available to them are different. The fact that our expressions for the loss turn out to be quite simple suggests that these could potentially play a role in the assessment of the incidence, across investors, of taxes on investment returns.

We also show that any given tax on investment returns is identical in its burden on investors to a proportional tax on their initial wealths. The latter tax rate depends solely on the former tax rate and the riskless return. Hence, the tax rate on the initial wealth is the same for all investors regardless of their characteristics (including their initial wealths) and the set of risky assets available to them. To our knowledge, such an analysis of tax-induced wealth-loss is unavailable in the literature.

The paper is organized as follows. Section II characterizes the wealth-loss to the investor using the concepts of equivalent and compensating variations. Section III presents the expressions for this loss, and also provides an intuition underlying them. Section IV concludes with some extensions of the preceding analysis.[1]

II. CHARACTERIZATIONS OF THE LOSS

Unless stated otherwise, "return" means the net return, and a distinction is made, wherever it is necessary, between the pre-tax and the post-tax return from an asset. The pre-tax riskless return is denoted by x_0. It is presently assumed that $x_0 > 0$; other values of x_0 are considered later. The subscripts $i = 1$ to n represent the risky assets. None of these $n + 1$ assets is redundant. The pre-tax random return from the risky asset i is denoted by x_i. The random excess return from the risky asset i is $z_i \equiv x_i - x_0$. The investor's utility function is $u(\ )$, which is increasing and strictly

[1] Constantinides (1984), Stiglitz (1983) and others have proposed intertemporal trading strategies that an investor could use primarily to reduce his tax liability. The empirical findings of Poterba (1987) suggest that most investors do not use such strategies, perhaps because of market imperfections (for example, borrowing constraints and transactions costs). An examination of such strategies is not a part of this paper.

concave. The investor's initial wealth is y, where $y > 0$. The expectation operator is denoted by $E[\ \ ]$.

Let t denote the tax rate on the return from investments. It is assumed throughout that $t < 1$; that is, the sign of the pre-tax investment return is the same as that of the post-tax investment return. We presently assume that $t > 0$; negative values of t are examined later. Let $U(y,t)$ denote the optimal expected utility that the investor achieves in the post-tax regime. Then,

$$U(y,t) = \max_{b_1,\ldots,b_n} : E[u(y\{1 + (1-t)(x_0 + \textstyle\sum_{i=1}^{n} b_i z_i)\})], \tag{1}$$

where b_i, for $i = 1$ to n, denote the portfolio weights (hereafter referred to as "weights") on the risky assets in the post-tax regime. The implied weight on the riskless asset is $1 - \sum_{i=1}^{n} b_i$. Unless stated otherwise, it is assumed that full tax-loss offset applies, as in (1). It is also assumed that the optimization problems in the right-hand-sides of (1) and (2) below, have interior solutions; that is, solutions with finite weights. Uniqueness of solutions is not needed for the present analysis.[2]

Correspondingly, let $U(y,0)$ denote the optimal expected utility that the investor achieves in the pre-tax regime. Thus,

$$U(y,0) = \max_{a_1,\ldots,a_n} : E[u(y\{1 + x_0 + \Sigma a_i z_i\})], \tag{2}$$

where the weights on the risky assets in the pre-tax regime are denoted by a_i, for $i = 1$ to n. In the summations over the risky assets, the range of summation (from $i = 1$ to n) is suppressed in (2) and in the rest of the paper.

[2]The two-period model of asset allocation used here can be extended by employing a two-period additive utility function and incorporating the first-period consumption. In this case, a modified version of the results presented in this paper holds.

Let U_y and U_t denote the derivatives of $U(y,t)$ with respect to its first and second arguments. As is shown in the Appendix, (1) yields the following two expressions that are used in the analysis below:

$$U_y(y,t) = \{1 + (1-t)x_0\}E[u_Y], \text{ and} \tag{3}$$

$$U_t(y,t) = -yx_0E[u_Y], \tag{4}$$

where $u_Y > 0$ is the derivative of u with respect to its argument. These expressions, and the assumptions that $t < 1$ and $x_0 > 0$, imply that

$$U_y > 0, \text{ and} \tag{5}$$

$$U_t < 0. \tag{6}$$

That is, the optimal expected utility is increasing in the investor's initial wealth, and decreasing in the tax rate.

Since $t > 0$, it follows from (6) that

$$U(y,0) > U(y,t). \tag{7}$$

That is, the investor in the post-tax regime is worse-off than in the pre-tax regime.

We now define the analogs of equivalent and compensating variations respectively. Let ε denote the investor's wealth-loss according to equivalent variation. Then, ε is the amount that, if taken away from the investor's initial wealth in the pre-tax regime, will yield him the same optimal expected utility as what he will actually have in the post-tax regime. Thus, ε is defined by

$$U(y-\varepsilon,0) = U(y,t). \tag{8}$$

In other words, the investor is indifferent between: (i) giving up an amount ε of his initial wealth in the pre-tax regime, and (ii) being in the post-tax regime.[3]

Likewise, let ϕ denote the investor's wealth-loss according to compensating variation. Then, ϕ is the amount that, if added to the investor's initial wealth in the post-tax regime, will yield him the same optimal expected utility as what he will actually have in the pre-tax regime. Thus, ϕ is defined by

$$U(y + \phi, t) = U(y, 0). \tag{9}$$

That is, the investor is indifferent between: (i) receiving an addition ϕ to his initial wealth in the post-tax regime, and (ii) being in the pre-tax regime.

Note that we are adapting the standard definitions of equivalent and compensating variations (see Varian (1992, pp. 167–8)) for comparing the investor's welfare levels, which are measured here by his expected utility. The key feature of equivalent variation is that the investor's post-tax welfare level serves as the yardstick to calculate his gain or loss. The corresponding yardstick in the case of compensating variation is the investor's pre-tax welfare level. Further, in defining ε and ϕ in (8) and (9), it is not assumed that any transfer of initial wealth actually occurs to or from the investor. These variables have been defined solely to assess the money-metric changes in the investor's welfare level.

Since the investor in the post-tax regime is worse-off than in the pre-tax regime,

[3]In many situations, equivalent variation is characterized as the amount "given to" the individual, and not as the amount "taken from," as defined here. We do this purely for expositional ease. In any event, with a change of sign, one is identical to the other. An analogous comment applies to the definition, presented below, of the loss according to compensating variation.

we would expect that

$$\varepsilon > 0, \text{ and} \tag{10}$$

$$\phi > 0. \tag{11}$$

These expressions are easily verified. From (7) and (8), $U(y,0) > U(y-\varepsilon,0)$. Hence, (5) implies (10). Likewise, from (7) and (9), $U(y+\phi,t) > U(y,t)$. Hence, (5) implies (11). Further, it is easily ascertained from (5), (8), and (9) that, for any given set of parameters, ε has a unique value, and so does ϕ.

III. EXPRESSIONS FOR THE LOSS

It is shown in the Appendix that

$$\varepsilon = yx_0t/(1+x_0), \text{ and} \tag{12}$$

$$\phi = yx_0t/\{1+(1-t)x_0\}. \tag{13}$$

It follows from these expressions that the investor's wealth-loss due to taxation, whether assessed according to equivalent or compensating variation, does not depend on the nature of his risk-aversion or on any aspect of the set of risky assets available to him. Further, the interpretation of the expressions for (12) and (13) is straightforward. Suppose that the investor's entire initial wealth was invested in the riskless asset. Then, the tax amount on the resulting gain, discounted at the pre-tax riskless return, is ε. The same tax amount, when discounted at the post-tax riskless return, is ϕ.

It is perhaps useful to note the following aspect of the expressions (12) and (13) for ε and ϕ. It is not assumed here that the investor invests his entire wealth in the riskless asset. On the contrary, the investor optimally allocates his wealth across assets in, respectively, the pre-tax and the post-tax regimes. As noted before, ε and ϕ are defined independently of the investor's behavior, only to assess the magnitude of the investor's loss.

The next three propositions follow from (12) and (13):

Proposition 1. *Suppose hypothetically that the investor's entire initial wealth was invested in the riskless asset. Then, the tax amount on the resulting gain, discounted at the pre-tax riskless return, is the loss of the investor's initial wealth due to the tax, when this loss is assessed according to equivalent variation.*

Proposition 2. *Suppose hypothetically that the investor's entire initial wealth was invested in the riskless asset. Then, the tax amount on the resulting gain, discounted at the post-tax riskless return, is the loss of the investor's initial wealth due to the tax, when this loss is assessed according to compensating variation.*

Proposition 3. *The loss of the investor's initial wealth due to the tax, whether this loss is assessed according to equivalent or compensating variation, is increasing in the pre-tax riskless return* x_0*. Further, as one would expect, the wealth-loss just noted is increasing in the tax rate* t*.*

An intuitive explanation of ε and ϕ. Recalling (12) and (13), it is apparent that these expressions for the investor's loss are highly parsimonious in their informational requirements. The only information that is needed to calculate them consists of the investor's initial wealth, the riskless return, and the tax rate. Moreover, both of these expressions for the initial wealth-loss are proportional to the investor's initial wealth. Hence, the losses can be expressed as percentages of the investor's initial wealth.

At first glance, it might appear somewhat surprising that ε and ϕ do not depend on the nature of the investor's risk-aversion or on any aspect of the set of risky assets available to him. Hence, we present here an intuition underlying ε; an analogous intuition applies to ϕ. For expositional simplicity, assume that there is only one risky asset (that is, $n = 1$); the reasoning is similar for the case of many risky assets. Then, as is well understood in portfolio analysis, the investment allocation of an investor can be viewed as consisting of two parts: (i) the entire initial wealth is invested in the riskless asset, and (ii) an amount (which can be positive, zero, or negative) is borrowed at the riskless rate and is invested in the risky asset. On the latter amount, the investor receives the random excess return. Hence, for brevity, we refer to this amount as the investor's "investment on the excess return." The investor achieves his maximum expected utility by choosing the optimal level of investment on the excess return. This standard way of describing the portfolio optimization problem can be seen in the right-hand-sides of (1) and (2).

For use below, recall that in the post-tax regime, the investor has an initial wealth y, he faces the tax rate t, and his optimal expected utility is $U(y, t)$, described in (1). We refer to the preceding utility as "the post-tax utility." We now show that if ε is subtracted from the investor's initial wealth y in the pre-tax regime, then the optimal expected utility that he achieves equals the post-tax utility.

To see this, note that this investor's initial wealth is $y - \varepsilon$, and he faces no tax. Hence, the terminal payoff from the first part of his allocation will be the same as that from the corresponding part in the post-tax regime. This is because, from (12), $(y-\varepsilon)(1+x_0) = y(1+(1-t)x_0)$. It follows immediately that all of the parameters that this investor faces are exactly the same as those that he faces in the post-tax regime. Hence, it will be optimal for him to choose the level of investment on the excess return which is the same as the corresponding level in the post-tax regime. The resulting optimal expected utility of this investor will thus be the same as his post-tax utility.

A numerical example. Suppose that the tax rate is 20% (that is, $t = 0.2$) and the investment horizon is one year. Suppose that the pre-tax riskless return is 3.7% (that is, $x_0 = 0.037$), which is the annualized return on the U. S. T-Bills from 1926 to 1995; see Ibbotson Associates (1996, p. 292). Then, from (12), the tax-induced loss is 0.714% of the initial wealth, when this loss is assessed according to equivalent variation. From (13), the corresponding loss is 0.719% of the initial wealth, when this loss is assessed according to compensating variation.

A tax on initial wealth that is identical in its welfare effect to a tax on investment returns. Rewrite the left-hand-side of (8) as $U(y(1-T), 0)$, where $T \equiv \varepsilon/y$. Then, it follows from (8) that a tax on the initial wealth at the rate T in the pre-tax regime leads to the same expected utility of the investor as that when a tax rate of t is imposed on his investment returns.[4] Further, from (12), it is obvious that the tax rate T depends solely on t and x_0. Hence, if a group of investors faces a tax rate of t on their investment returns, then this tax regime is identical in its

[4]In addition, suppose that the optimization problems in the right-hand-sides of (1) and (2) have unique interior solutions. Then, it can be verified that the investor's terminal payoffs from the riskless asset as well as each of the risky assets are the same under these two alternative tax regimes.

welfare effect on each of the investors to a tax on their initial wealths at the rate of T, even if these investors differ in their characteristics (including their initial wealths) and the set of risky assets available to them.

IV. REMARKS AND EXTENSIONS

Some extensions are presented in this section. Each of these is considered separately from the others. In each subsection below, the assumptions of the previous sections are maintained, unless stated otherwise.

Pre-existing taxes. In the preceding analysis, the investor faces a change in the tax rate from zero to t. The approach developed there can however be easily modified to deal with a pre-existing tax. Suppose that τ is the tax rate on investment returns in the "pre-change" regime, where $0 \leq \tau < 1$, and t is the rate in the "post-change" regime. Then, once again, we can obtain the expressions for the investor's loss of initial wealth (which could be negative) from this change, according to equivalent and compensating variations respectively.

Let $\varepsilon(\tau, t)$ be the loss according to equivalent variation, and $\phi(\tau, t)$ be the loss according to compensating variation. These are defined, analogous to (8) and (9), by

$$\begin{aligned} &U(y - \varepsilon(\tau, t), \tau) = U(y, t), \text{ and} \\ &U(y + \phi(\tau, t), t) = U(y, \tau). \end{aligned} \tag{14}$$

Applying reasoning analogous to that used earlier, it can be shown that

$$\begin{aligned} &\varepsilon(\tau, t) = yx_0(t - \tau)/\{1 + (1 - \tau)x_0\}, \text{ and} \\ &\phi(\tau, t) = yx_0(t - \tau)/\{1 - (1 - t)x_0\}. \end{aligned} \tag{15}$$

As before, these expressions are intuitive and informationally parsimonious. If the investor faces an increased tax rate (that is, $0 < \tau < t$), then, from (15), the investor becomes worse-off in the post-change regime. The opposite is true if the investor faces a decreased tax rate; that is, if $0 < t < \tau$. The results of the previous section are obtained if $\tau = 0$.

Subsidies. If $t < 0$, then, from (6), the inequality (7) is reversed. That is, the investor in the post-tax regime is better-off than in the pre-tax regime. It is then easily ascertained that $\varepsilon < 0$, and $\phi < 0$, and that the expressions (12) and (13) continue to hold.

Riskless return. If the riskless return is zero, then (4) implies that the investor's expected utility remains unchanged due to taxation. This result has been established, in a context altogether different from that of the present paper, by Sandmo (1977, 1989).

If the riskless return is negative, then from (4), a positive tax rate implies that the investor in the post-tax regime is better-off than in the pre-tax regime. The inequality (5) continues to apply in the present case, because it is appropriate to assume in (3) that $x_0 > -1$; that is, the gross pre-tax riskless return is positive. Once again, it can be established that the conclusions presented in the last sentence in the previous subsection hold.

Restrictions on the tax-loss offset. In several countries, an investor's tax gain on his investment loss is not allowed to be offset against his other tax liabilities, especially if the investment loss exceeds some pre-defined non-negative amount. In some countries, the investor can carry the unused tax gain for future offsets, in which case the value of the tax gain is lower than what it would have been if it did not need to be carried forward.

The investor is better-off with full tax-loss offset than with restrictions on the tax-loss offset. It thus follows that, in the latter case, ε and ϕ are lower bounds of the investor's loss of initial wealth.

APPENDIX

Derivation of (3). The first-order conditions for the optimality of (1) are

$$E[z_k u_Y] = 0, \text{ for } k = 1 \text{ to } n. \tag{A1}$$

Using the envelope theorem, (1) yields $U_y(y,t) = E[\{1 + (1-t)(x_0 + \Sigma b_i z_i)\}u_Y] = \{1 + (1-t)x_0\}E[u_Y] + (1-t)\Sigma b_i E[z_i u_Y]$. From (A1), the second term in the preceding expression is zero. Hence,

$$U_y(y,t) = \{1 + (1-t)x_0\}E[u_Y]. \tag{A2}$$

Derivation of (4). The use of the envelope theorem on (1) yields $U_t(y,t) = -E[y(x_0 + \Sigma b_i z_i)u_Y] = -yx_0 E[u_Y] - y\Sigma b_i E[z_i u_Y]$. From (A1), the second term in the preceding expression is zero. Hence,

$$U_t(y,t) = -yx_0 E[u_Y]. \tag{A3}$$

Derivation of (12). Recall that (8) defines a unique value of ε. Hence, if (12) implies (8), which is what we show here, then it follows that (8) implies (12). Analogous to (2),

$$U(y-\varepsilon, 0) = \max_{\beta_1,\ldots,\beta_n} : E[u((y-\varepsilon)\{1 + x_0 + \Sigma\beta_i z_i\})]. \tag{A4}$$

Using (12),

$$(y-\varepsilon)(1+x_0) = y(1+(1-t)x_0). \tag{A5}$$

Next, define B_i such that

$$(y-\varepsilon)\beta_i = y(1-t)B_i. \tag{A6}$$

For the relationship (A6) to be non-degenerate, it is necessary that $y - \varepsilon \neq 0$. The preceding condition is satisfied because: (i) from (A5), $y - \varepsilon = y(1 + (1 - t)x_0)/(1 + x_0)$, and (ii) $t < 1$ and $x_0 > 0$. Substitution of (A5) and (A6) into (A4) yields

$$U(y - \varepsilon, 0) = \max_{B_1,\ldots,B_n} : E[u(y\{1 + (1 - t)(x_0 + \Sigma B_i z_i)\})]. \tag{A7}$$

The right-hand-sides of (1) and (A7) are equal. Hence, the left-hand-sides of (1) and (A7) must also be equal. This establishes (8).

Derivation of (13). This derivation is analogous to that of (12). Recalling (1),

$$U(y + \phi, t) = \max_{\alpha_1,\ldots,\alpha_n} : E[u((y + \phi)\{1 + (1 - t)(x_0 + \Sigma \alpha_i z_i)\})]. \tag{A8}$$

Using (13),

$$(y + \phi)(1 + (1 - t)x_0) = y(1 + x_0). \tag{A9}$$

Next, define A_i such that

$$(y + \phi)(1 - t)\alpha_i = yA_i. \tag{A10}$$

For the relationship (A10) to be non-degenerate, it is necessary that $y + \phi \neq 0$. The preceding condition is satisfied because: (i) from (A9), $y + \phi = y(1 + x_0)/(1 + (1 - t)x_0)$, and (ii) $t < 1$ and $x_0 > 0$. Substitution of (A9) and (A10) into (A8) yields

$$U(y + \phi, t) = \max_{A_1,\ldots,A_n} : E[u(y\{1 + x_0 + \Sigma A_i z_i\})]. \tag{A11}$$

The right-hand-sides of (2) and (A11) are equal. Hence, the left-hand-sides of (2) and (A11) must also be equal. This establishes (9).

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